Central Kähler Metrics with Non-Constant Central Curvature[*]

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Abstract

The *central curvature* of a Riemannian metric is the determinant of its Ricci endomorphism, while the scalar curvature is its trace. A Kähler metric is called *central* if the gradient of its central curvature is a holomorphic vector field [Ms2]. Such metrics may be viewed as analogs of the extremal Kähler metrics defined by Calabi [C1, C2]. In this work, central metrics of non-constant curvature are constructed on various ruled surfaces, most notably the first Hirzebruch surface. This is achieved via the momentum construction [HSn], a variant of an ansatz employed in the works [C0, KS1]. Non-existence, real-analyticity and positivity properties of central metrics arising in this ansatz are also established.

1 Introduction

Central Kähler metrics were investigated in [Ms2], where an extensive comparison was carried out with the extremal Kähler metrics of Calabi [C1, C2]. The latter are defined in terms of the scalar curvature, the former via the central curvature. Both are elementary symmetric functions of the Ricci endomorphism: The scalar curvature is its trace, while the central curvature is its determinant. The corresponding metric notion relates each curvature potential with the complex-analytic structure of the underlying manifold.

The main achievement of this work is the construction of central Kähler metrics with non-constant central curvature on various ruled surfaces, using the momentum construction, following [HSn]. Under appropriate geometric hypotheses, this ansatz converts the construction of a metric into a boundary value problem in one variable. The ansatz is equivalent to the one introduced by Calabi [C0, C1], but is expressed in coordinates more convenient for present purposes. It should be noted that the boundary value problem for

central metrics is more complicated than that arising for extremal ones, where a completely explicit solution may be obtained.

The spaces on which the momentum construction is defined include manifolds of the form $N = \mathbf{P}(L \oplus \mathcal{O})$, where $p : L \to M$ is a holomorphic Hermitian line bundle over a compact Kähler base manifold M, as well as the spaces obtained from such a ruled manifold by blowing down one or both of its ends to a lower dimensional submanifold. In other words, the ansatz is set-up on S^1-equivariant compactifications of C^*-bundles. This paper contains structure theorems for central metrics in this general setting, and existence-related theorems in the case where M is a compact Riemann surface.

Describing the latter first, let k be the Euler number of the Riemann surface, and $n > 0$ the Chern number of L. Our results suggest that the ansatz yields (non-constant) central metrics exactly when $n < |k|$, with nowhere vanishing central curvature. This is proven for every n when the Euler number k vanishes, i.e. when the base is a one-dimensional torus. For other values of the Euler number, special cases are proven. The following theorem summarizes what is demonstrated, with regards to existence as well as non-existence:

Theorem 1.1 *The momentum construction yields a unique isometry class of real-analytic central metrics with non-constant central curvature, in every Kähler class of the following spaces:*

○ *The first Hirzebruch surface $\mathbf{P}(\mathcal{O}(1) \oplus \mathcal{O}) \to \mathbf{P}^1$, with the metrics having positive central curvature and a positive definite Ricci tensor.*

○ *Any ruled surface $\mathbf{P}(L \oplus \mathcal{O}) \to \Sigma_2$, where L has Chern number one, and Σ_2 is a compact Riemann surface of genus two, with the metrics having negative central curvature.*

The momentum construction yields no central metrics on the second Hirzebruch surface, and none with non-constant central curvature on any ruled surface $\mathbf{P}(L \oplus \mathcal{O})$ over a compact Riemann surface of genus one.

Our methods also work for other special choices of Chern number and Kähler class, and it is hoped that they can be extended to higher dimensional Kähler-Einstein base spaces as well.

Turning to the structure theorems that hold in the above set-up, we note that the (non-constant) central metrics of the momentum construction are never extremal, and also demonstrate that if the base manifold is Fano, the central curvature of g is positive if and only if its Ricci tensor is positive definite. Furthermore, there is a canonical local representation of the metric along a fiber, with respect to which it has a base and a vertical component. We show the following:

Theorem 1.2 *If the vertical component of a central metric is of class C^2, then it is real-analytic.*

See Theorem 3.6 for the precise statement. It follows from severe restrictions implied by centrality on the functions used in the construction of the solution metric. These also imply further non-existence results for a variety of higher-dimensional base spaces.

Section 2 summarizes the pertinent features of the momentum construction; details can be found in [HSn]. Earlier forms of it appear in [C0, KS1, KS2]. Many other works make use of the ansatz to construct metrics with various curvature properties, including [Ki, Sk, H3, G, Mb2, Sm, LSn, PP, Tf]. In Section 3 the central curvature is expressed in terms of the data used in the ansatz. Boundary and integrability conditions for the resulting centrality equation, as well as the highly restrictive properties of the latter are then pursued. Theorem 1.2, positivity, and non-extremality then follow from these considerations.

Section 4 is devoted to the proof of Theorem 1.1. After reviewing the ansatz for the case of a one-dimensional base, branching possibilities for real-analytic functions used in the construction of possible solution metrics are classified according to the topology. The final sections detail uniqueness and existence possibilities as deduced from the solvability of a pivotal non-linear integrability condition.

The definition of a central Kähler metric below concludes this introduction. Let (N, g) be a compact Kähler manifold of complex dimension ℓ. Denoting its Kähler form by ω and its Ricci form by ρ, the central curvature function C is defined by the equation

$$\rho^{\wedge \ell} = C \omega^{\wedge \ell}.$$

In other words, if one defines the Ricci endomorphism ϱ via the metric (by 'raising the second index' of ρ with respect to ω), then the central curvature is simply the complex determinant $\det \varrho$. A *central Kähler metric* (of type *I*, in the terminology of [Ms2]) is a metric for which the $(1,0)$-gradient vector field of C is holomorphic.

2 Salient Features of the ansatz

First, to describe the manifolds N considered in this work we give the following brief review of the essential ingredients of the momentum construction. Its basic data are a holomorphic, Hermitian line bundle $p : (L, h) \to (M, \omega_M)$ over a compact connected Kähler manifold, and an interval of real numbers that is hereby assumed to be of the form $[-T, T]$ for some $T > 0$ (sufficiently small). The curvature form of (L, h) is denoted γ, and the related ω_M-associated endomorphism of $T^{1,0}M$ is denoted B. The manifold N associated to these data is now taken to be either $\widehat{L} := \mathbf{P}(L \oplus \mathbf{1}) \xrightarrow{p} M$ — the completion of the total space of L — or else the space obtained from $\widehat{L}$ (if possible) by blowing down the zero-section, the infinity-section, or both, to submanifolds of higher codimension. Describing a space N of the latter type in some more detail, there are natural identifications of M with the zero- and infinity sections of $\widehat{L}$, and if D_0 and D_∞ denote the corresponding centers of blowing up in N, of respective complex codimensions d_0 and d_∞, then there are projections from M to D_0 and D_∞. These may be constructed from a pair of holomorphic vector

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bundles $E_0 \to D_0$ and $E_\infty \to D_\infty$ for which $\mathbf{P}(E_0) \simeq \mathbf{P}(E_\infty)$, together with respective dual tautological bundles. Non-trivial examples are given in [KS2, FMS].

On N, a Kähler class Ω is constructed out of the momentum data by requiring that its restriction to the zero section of N is $[\omega_M - T\,\gamma]$ and that the $\mathbf{P}^1$ fibres of N (or their images under the blow-down map) have area $4\pi T$. Consequently, the restriction of Ω to the infinity section of N is $[\omega_M + T\,\gamma]$.

Remark 2.1 When $N = \widehat{L}$, it is *assumed* that the closed $(1,1)$-forms $\omega_M \pm T\,\gamma$ are positive. Otherwise they are required to be "suitably" non-negative on M, in the sense that $\omega_M + T\,\gamma$ is the pullback of a Kähler form on D_0 and $\omega_M - T\,\gamma$ is the pullback of a Kähler form on D_∞. In terms of the curvature endomorphism B, suitable non-negativity means that the endomorphism $I - \tau B$ has nullity $d_0 - 1$ when $\tau = -T$, and nullity $d_\infty - 1$ when $\tau = T$. $\square$

Given the data above, the momentum construction associates a smooth Kähler metric g_φ on N to each smooth ("profile") function $\varphi : [-T, T] \to \mathbf{R}$ that is positive on $(-T, T)$ and satisfies the boundary conditions

$$\varphi(\pm T) = 0, \qquad \varphi'(\pm T) = \mp 2. \tag{2.1}$$

The associated Kähler form, denoted $\omega_\varphi = \omega$, represents Ω. The isometry class of g_φ is determined by its restrictions to the zero- and infinity sections, and by the profile φ. Distinct profiles do in fact yield distinct metrics.

Before describing the metric g_φ and its geometry in more detail, we list further curvature specifications of the ansatz that reduce curvature determination of the metric to an ordinary differential equation for φ. The base metric ω_M and the fibre metric h are assumed to satisfy the following conditions:

1. The eigenvalues of the curvature endomorphism B are constant on M.

2. The eigenvalues of the Ricci endomorphism ϱ_M are constant on M.

3. The endomorphisms B and ϱ_M are pointwise simultaneously diagonalizable.

The Metric and its Geometry

Since single-direction variations of the Kähler form ω_M will be considered in Section 4, it will be convenient to parameterize these by viewing ω_M as a multiple of *some* fixed Kähler form ω_0. Thus we set $\omega_M = 2\pi a \omega_0$, for some positive real number a.

The following equation on top-dimensional forms in M defines a function Q on $[-T, T]$ via

$$(\omega_M - x\,\gamma)^{\wedge(\ell-1)} = \frac{1}{a^{\ell-1}} Q(x)\,\omega_M^{\wedge(\ell-1)}.$$

In other words, $Q(x) = a^{\ell-1} \det(I - xB)$. Under the curvature assumptions made above, Q is a non-negative *polynomial* function of degree at most $\ell-1$. It is non-vanishing in $(-T, T)$,

and vanishes to order $d_0 - 1$ at $-T$ and to order $d_\infty - 1$ at T. In particular, if $N = \widehat{L}$, then Q is positive in $[-T, T]$. The Kähler forms $\omega_M - x\,\gamma$ all have the same Ricci form, denoted ρ_M. It is also useful to introduce the function

$$\Psi := \frac{1}{2Q}(\varphi Q)' = \frac{1}{2}\left(\varphi' + \frac{\varphi}{Q}\,Q'\right).$$

The latter is smooth since φ and Q are, and since its zeros cancel with its singularities at $x = \pm T$. By (2.1), it satisfies the boundary conditions

$$\Psi(-T) = d_0, \qquad \Psi(T) = -d_\infty. \tag{2.2}$$

The open set $N^\times := N \setminus (D_0 \cup D_\infty)$ is the total space of a $\mathbf{C}^\times$-bundle over M. At each point $z \in M$ there exist holomorphic coordinates $\{z^\alpha\}$ such that $z^\alpha(z) = 0$ for all α and the components of ω and its Ricci form $\rho := \rho_\varphi$ *along the fibre over z* are given by the block matrices

$$[g_{i\bar{j}}] = \begin{bmatrix} \frac{1}{\varphi(\tau)} & 0 \\ 0 & (\omega_M - \tau\,\gamma)_{\alpha\bar{\beta}} \end{bmatrix}; \qquad [\rho_{i\bar{j}}] = \begin{bmatrix} -\frac{1}{\varphi(\tau)}\Psi'(\tau) & 0 \\ 0 & \left(\rho_M + \Psi(\tau)\,\gamma\right)_{\alpha\bar{\beta}} \end{bmatrix}. \tag{2.3}$$

Here Greek indices run through $1 \ldots \ell - 1$, while the Roman ones run through $0 \ldots \ell - 1$. One can regard τ as the identity function on $(-T, T)$, with the above matrix-valued functions considered as living on $(-T, T) \times M$. A more direct description on N is obtained from the following geometric presentation of the ansatz.

There is a holomorphic circle action on N, induced by scalar multiplication in the fibres of L. The moment map with respect to ω is a real-valued function τ whose image is $[-T, T]$, and the function $\varphi(\tau)^\dagger$ is the length (squared) of the standard generator of the circle action. The gradient of τ with respect to ω is induced from the Euler field on $\widehat{L}$. Note that holomorphy potentials that depend solely on τ are necessarily τ-affine functions.

The symplectic reduction of N at $\{\tau = x\}$ is $(M, \omega_M - x\,\gamma)$; it is often helpful to view $\omega_M - x\,\gamma$ as a (non-closed) form on $[-T, T] \times M$. This is in line with the positivity conditions assumed for $\omega_M \pm T\,\gamma$. Projecting N to this space gives, over each point $(x, z) \in [-T, T] \times M$ with $-T < x < T$, a fiber which is a circle of length $2\pi\sqrt{\varphi(x)}$. Together with the positivity of φ, this accounts for the first pair of boundary conditions in (2.1). The second pair ensures the smooth extendibility of the metric to the zero and infinity sections. Finally, up to a constant multiple, $Q(\tau)\,d\tau$ is the induced Duistermaat-Heckmann measure on $[-T, T]$, and the function $\varphi Q(\tau)$ is the volume density of g_φ on N.

Remark 2.2 Note that the block-diagonality of the relations (2.3) holds only along a single fibre, with the off-diagonal terms eliminated by a particular choice of gauge; see [KS1] or [HSn] for the complete expressions. However, due to the algebraic nature of the central curvature, these relations are sufficient for its calculation. $\qquad\qquad\square$

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†Actually $\varphi \circ \tau$, but the notation $\varphi(\tau)$ is compelling and useful.

Example 2.3 In the prototypical example, M is a compact Kähler-Einstein manifold of dimension $\ell - 1$ (or even a product of such spaces of varying dimensions), with ω_0 now an indivisible and integral Kähler form which is a multiple of the first Chern class of M. Let a and T be positive real numbers, and n a positive integer such that $a - nT > 0$. The holomorphic Hermitian line bundle (L, h) is chosen to have curvature form $\gamma = 2\pi n \omega_0$. Such a bundle always exists, and is unique if M is simply-connected. The base metric is still $\omega_M = 2\pi a \omega_0$, and its Ricci form is $\rho_M = 2\pi k \omega_0$ for some integer k. The curvature and Ricci endomorphisms are multiples of the identity, with eigenvalues n/a and k/a respectively. The metric ω lives on the completion of L, and a quadratic profile satisfying (2.1) provides a simple example for which the $\mathbf{P}^1$ fibres of the induced metric are round spheres. $\qquad\square$

3   Central Metrics in the Momentum Construction

Fix a holomorphic Hermitian line bundle $p : (L, h) \to (M, \omega_M)$ and a momentum interval $[-T, T]$ as in Section 2. Let η_i and κ_i ($1 \leq i \leq \ell - 1$) be the constant, not necessarily distinct eigenvalues of the curvature endomorphism B and of Ricci endomorphism ϱ_M, respectively, indexed with respect to a simultaneously diagonalizing eigenframe $\{e_i\}$.

3.1   The Central Curvature

The polynomial Q is given by

$$Q(x) = a^{\ell-1} \prod_{i=1}^{\ell-1} (1 - \eta_i \, x); \tag{3.1}$$

all its roots are real, exactly $d_0 - 1$ linear factors vanish at $-T$, and $d_\infty - 1$ vanish at T. It will be of great use to introduce another polynomial function on $[-T, T]$:

$$R'(x) := a^{\ell-1} \prod_{i=1}^{\ell-1} (\kappa_i + \eta_i \, x). \tag{3.2}$$

For each x it is the determinant of the form $\rho_M + x\gamma$, regarded as an endomorphism via the Kähler form ω_M. Its antiderivative will be of equal interest to us, as indicated by the prime notation. The polynomials Q and R' obviously have the same degree, equal to the rank of the curvature endomorphism B.

The central curvature $C = C(\tau)$ may be specified in the momentum construction. It is immediately calculable along a fibre of $N^\times$ from relations (2.3):

$$C(\tau) = \varphi \left(-\frac{1}{\varphi} \Psi'(\tau) \right) \prod_{i=1}^{k} \left(\frac{1}{1 - \tau\eta_i} \left(\kappa_i + \Psi(\tau)\eta_i \right) \right).$$

In terms of the functions just defined, it is given by

$$C(\tau) = -\frac{1}{Q(\tau)}R'\big(\Psi(\tau)\big)\Psi'(\tau) = -\frac{(R \circ \Psi)'}{Q}(\tau). \tag{3.3}$$

Because the central curvature is a scalar, and thus independent of the choice of the coordinate system, this equation holds globally on $N^\times$, hence on N by continuity. The central curvature is "a function of τ alone", so, as noted previously, it is a holomorphy potential if and only if it is affine in τ.

A priori, the right-hand side of (3.3) has poles where Q vanishes. However, Lemma 3.1 below asserts that endpoint roots of Q correspond to roots of $R' \circ \Psi$. This is as it should be, since an arbitrary Kähler metric on N *has* a smooth central curvature function.

Lemma 3.1 *With notations as above, if $1 + \eta_i\,T = 0$, then $\kappa_i + \eta_i\,d_0 = 0$. Similarly, if $1 - \eta_i\,T = 0$, then $\kappa_i - \eta_i\,d_\infty = 0$. More succinctly, if a linear factor of Q vanishes at an endpoint $\pm T$, then the corresponding linear factor of $R' \circ \Psi$ also vanishes at that endpoint.*

Proof   In $T^{1,0}M$, choose the local frame $\{e_i\}_{i=1}^{\ell-1}$ so that the first $d_0 - 1$ vectors are annihilated by the projection $M \to D_0$. The vertical tangent bundle of this projection is exactly the null eigenbundle of $I + TB$, so if $1 + \eta_i T = 0$ for some i, then D_0 has codimension greater than 1, hence $M \to D_0$ is a non-trivial fibration. Its fibres are projective spaces of dimension $d_0 - 1$. The line bundle $L \to M$ is the tautological bundle of the normal bundle of D_0 in N, so its restriction to a fibre F of $M \to D_0$ is $\mathcal{O}_F(-1)$. Therefore, if the restriction $\omega_M|_F$ is $\hat{a}$ times the unit Fubini-Study form on F, then

$$\eta_i = -\frac{1}{\hat{a}} \qquad \text{for } 1 \le i \le d_0 - 1.$$

Since a fiber F has anticanonical line bundle $\mathcal{O}(d_0)$,

$$\kappa_i = \frac{d_0}{\hat{a}} \qquad \text{for } 1 \le i \le d_0 - 1.$$

The lemma now follows by immediate substitution. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

For brevity, the polynomial R' is called "*essentially non-vanishing*" if its only roots in the closed interval $[-d_\infty, d_0]$ are those accounted for by Lemma 3.1. If the image of Ψ is equal to $[-d_\infty, d_0]$, then R' is essentially non-vanishing if and only if $(R' \circ \Psi)/Q$ is non-vanishing on the *closed* interval $[-T, T]$.

3.2   Integrability Conditions

Given a smooth profile function with the correct non-negativity behavior, construction of a central metric in this ansatz amounts to solving a second-order differential equation in φ, obtained by setting the central curvature to be affine

$$C(\tau) = b_0 + b_1\tau,$$

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and subject to the four boundary conditions (2.1). Using Ψ and equation (3.3), this can be expressed as a system of two first-order equations in φ and Ψ

$$b_0 + b_1\tau = -\frac{1}{Q(\tau)}(R \circ \Psi)'(\tau), \qquad \Psi = \frac{1}{2Q}(\varphi Q)', \tag{3.4}$$

subject to the two boundary conditions (2.2), and to two further integrability conditions that will presently be described. Whenever the equation can be solved, these conditions are expected to determine the coefficients b_0 and b_1 uniquely.

Multiplying (3.3) by Q and integrating from $-T$ to τ, gives

$$P(\tau) := -\int_{-T}^{\tau} C(x)Q(x)\,dx = R \circ \Psi(\tau) - R(d_0), \tag{3.5}$$

for an arbitrary smooth function C. The function P will play an important role in what follows. It is polynomial whenever C is. In particular, if g is a central metric of non-constant central curvature, P has degree equal to $2 + \deg Q$.

Setting $\tau = T$ gives the first integrability condition

$$R(-d_\infty) - R(d_0) = \int_{d_0}^{-d_\infty} R' = -\int_{-T}^{T} CQ. \tag{3.6}$$

This gives a boundary condition on an arbitrary profile in terms of an integral involving the central curvature. For a central metric, equation (3.6) gives a linear condition on the constants b_0 and b_1 and will thus be referred to as the linear integrability condition. Since $P(-T) = 0$, choosing $R(d_0) = 0$ simplifies condition (3.6) to

$$R(-d_\infty) = P(T). \tag{3.7}$$

For a central metric the right-hand side integral is indeed linear in b_0 and b_1, which leads to the possibility of eliminating one of these coefficients.

To get a second integrability condition, assume first that R' is non-vanishing on the image of Ψ (except possibly at the endpoints). It follows that there is a single branch of R^{-1} on the image of Ψ. Hence, still using the choice $R(d_0) = 0$, one has $\Psi = R^{-1} \circ P$ on $[-T, T]$. By the definition of Ψ and the boundary conditions (2.1) on φ,

$$2\int_{-T}^{T} Q\,\Psi = (\varphi Q)(T) - (\varphi Q)(-T) = 0.$$

Substituting for Ψ gives the second integrability condition:

$$\int_{-T}^{T} Q \cdot (R^{-1} \circ P) = 0. \tag{3.8}$$

This condition again holds for arbitrary profile functions. When the metric g is central, it will later be shown that it is not only smooth but in fact (vertically) real-analytic, and that

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the domain of R can be split into at most two intervals on which R is monotone. For the case of two intervals, R^{-1} is composed of two branches R_I^{-1} and R_{II}^{-1} that patch smoothly at a point $\tau_0 \in (-1, 1)$, and then (3.8) reads

$$\int_{-T}^{\tau_0} Q \cdot (R_I^{-1} \circ P) + \int_{\tau_0}^{T} Q \cdot (R_{II}^{-1} \circ P) = 0. \tag{3.9}$$

The point τ_0 will be characterized in the next two subsections as the unique one for which $(R' \circ \Psi)(\tau_0) = 0$.

Since equation (3.8) involves the inversion of R, it is relatively difficult to analyze. The appearance of R^{-1} prevents it, in general, from being linear in b_0 and b_1. Henceforth it will be referred to as the non-linear integrability condition.

3.3 Implications of the Linearity of the Central Curvature

So far the metrics arising from the momentum construction have been assumed to be smooth. Unless the profile function is given explicitly, this may be difficult to verify in general. It will thus be useful to broaden the scope of allowed profiles. Assume N admits a non-smooth central Kähler metric g_φ for which φ is of class C^2. With such assumptions equation (3.4) is still well-defined, and since $C(\tau)$ is affine, implies the following:

Proposition 3.2 *For any central metric arising from the momentum construction with a profile of class C^2 and with central curvature not identically zero, exactly one of the following possibilities holds.*

○ *The central curvature and Ψ' do not vanish anywhere in $[-T, T]$, and R' is essentially non-vanishing in $(-d_\infty, d_0)$.*

○ *The central curvature and Ψ' both vanish without multiplicity on the same unique value of τ in $[-T, T]$, and R' is essentially non-vanishing on $[-d_\infty, d_0]$ (or even on the image of Ψ).*

○ *The central curvature vanishes for a unique value of τ in $[-T, T]$, R' has a unique linear factor with a zero in $[d_\infty, d_0]$ besides possibly those vanishing at the endpoints by Lemma 3.1, and Ψ' is nowhere vanishing on $[-T, T]$.*

It is implicitly understood above that for non-smooth functions, multiplicity of vanishing is defined *ad hoc* to be one more than the number of well-defined vanishing derivatives at the point.

Remark 3.3 Note that since in any case $R' \circ \Psi$ vanishes in at most one point of $(-T, T)$, this immediately rules out existence of central metrics on many spaces. In other words, by the third possibility above, if R' has more than one root in $(-d_\infty, d_0)$, then N does not admit a central metric arising from the momentum construction. This happens, for

example, if M is a Ricci-flat manifold of dimension at least two and L is a line bundle with non-degenerate curvature. □

Remark 3.4 Since our interest lies in *non-constant* central curvature, it was assumed above that it does not vanish identically by excluding $R' \equiv 0$. The latter leads to a product metric, with L a flat line bundle and M a base also of vanishing central curvature. The other possibility for $C = 0$, namely $\Psi \equiv c$ on $[-T, T]$, cannot occur. For otherwise, by definition of Ψ and the boundary conditions (2.1), $0 = (\varphi Q)\big|_{-T}^{T} = 2c \int_{-T}^{T} Q$, so $c = 0$. Thus φQ is constant, hence also identically zero. □

Remark 3.5 If Ψ' is nowhere vanishing in $(-T, T)$, Ψ decreases from d_0 to $-d_\infty$. In fact, that $\Psi' < 0$ in $(-T, T)$ follows from assuming that Ψ has image equal to $[-d_\infty, d_0]$, will become apparent after proving in the next subsection that centrality of the metric implies that it is (vertically) real-analytic. This indeed follows since Ψ would then be *real-analytic* with extrema at the endpoints and with at most one critical point (counting multiplicity), which is not an endpoint. Consequently Ψ' has no such zeros.

Again using real-analyticity, the second possibility of the proposition cannot occur at an interior point if neither end of N is a divisor: if $C(\tau)$ and Ψ' change sign exactly once at an interior point, this implies that Ψ increases near one of the endpoints and consequently has image larger than $[-d_\infty, d_0]$. By the intermediate value theorem, its value at that endpoint is attained in the *open* interval $(-T, T)$, and if R' vanished on it, as happens with lower dimensional ends, the allowed number of zeros of C would be exceeded. □

3.4   Smoothness Considerations for Central Metrics

This subsection is devoted to showing that centrality implies real-analyticity, at least for the vertical component of a metric arising from the momentum construction, even in a situation where the corresponding profile is not known to be smooth or non-negative.

In fact, as noted previously, as long as R is invertible, $\Psi = R^{-1} \circ P$ defines a real-analytic function on $[-T, T]$, and therefore if Ψ solves the centrality equations (3.4), φ and consequently the vertical part of g are also real-analytic. The following renders Theorem 1.2.

Theorem 3.6 *Let φ be a function of class C^2 satisfying the boundary conditions (2.1), and which (together with a related function Ψ) gives a solution to equations (3.4). If R' is not identically zero, then φ is real-analytic. In addition φ is positive on $(-T, T)$.*

Proof   As mentioned above, to prove φ is real-analytic, it is enough to show Ψ is. If R' is not identically zero, then its roots are discrete, and Proposition 3.2 implies that the image of Ψ can be partitioned into at most two intervals on which R' does not vanish, except perhaps at their endpoints. On each such interval, there is a continuous branch of R^{-1} (which is real-analytic except perhaps at the endpoints), so as before $\Psi = R^{-1} \circ P$ is real-analytic except possibly at a point τ_0 with $[(R' \circ \Psi)/Q](\tau_0) = 0$.

Since Ψ is of type C^1 by assumption, Ψ' is well-defined. If there is such a $\tau_0 \in (-1, 1)$, then the third possibility in Proposition 3.2 is satisfied, so that Ψ' is non-vanishing on $[-T, T]$. Write equation (3.3) using the inverse function theorem as

$$\Psi'(\tau) = -\frac{C(\tau)Q(\tau)}{(R' \circ \Psi)(\tau)} = \left((R^{-1})' \circ P\right)(\tau) \cdot P'(\tau).$$

The order of vanishing of CQ at τ_0 must then necessarily equal the order of vanishing of R' at $\Psi(\tau_0)$; call this number δ. By (3.5), there exist polynomials P_0 and R_0, not vanishing at τ_0 and $\Psi(\tau_0)$, respectively, such that

$$P(x) = P(\tau_0) + (x - \tau_0)^{\delta+1} P_0(x) \qquad R(y) = R\left(\Psi(\tau_0)\right) + \left(y - \Psi(\tau_0)\right)^{\delta+1} R_0(y).$$

Since the powers in these expressions are equal, there exist branches of R^{-1} that can be patched to make $R^{-1} \circ P$ real-analytic at τ_0, and those can thus be used to define Ψ on $[-T, T]$.

Regarding positivity, by definition $(\varphi Q)' = 2Q\Psi$, while $\Psi(-T) = d_0 > 0 > -d_\infty = \Psi(T)$ by the boundary conditions on φ. Using the real-analyticity, it follows that Ψ vanishes an odd number of times in $(-T, T)$, counting multiplicity. If Ψ had three or more zeros, then it would have at least two critical points, contradicting Proposition 3.2. Thus Ψ vanishes exactly once, so φQ has exactly one critical point. Since φ is positive near $\pm T$ and Q is positive on $(-T, T)$, the product is positive on $(-T, T)$, as desired. $\qquad\square$

3.5 Positivity Conditions

In this subsection the positivity of the central curvature is related with the positivity of $c_1(N)$. This situation is an example of the first possibility listed in Proposition 3.2. Note the following related though simpler conclusion: since 0 is a value of Ψ, the sign of the central curvature of g_φ is the same as the sign of $\prod_i \kappa_i$, the central curvature of g_M.

The following is the positivity result alluded to in the introduction, and should be considered a consequence of Proposition 3.2. It is conceivable that it may be proved without alluding to the particularities of the momentum construction.

Theorem 3.7 *Let N be a (possibly blown-down) ruled manifold as above, and assume N admits a central Kähler metric g arising from the momentum construction. The following are then equivalent:*

○ *The Ricci curvature of g is positive.*

○ *The base space M is Fano and the central curvature of g is positive.*

In particular, the latter conditions imply N is Fano.

Proof   Though positivity of the Ricci tensor obviously implies positivity of the central curvature, it is instructive to prove this in the framework of the ansatz.

Assume the Ricci curvature of g is positive. Then by the Kähler and Ricci form expressions (2.3), $\Psi' < 0$ on $(-T, T)$, since φ is positive there. Turning to the endpoints, the ratio of the vertical components of the Ricci and Kähler forms is $-\Psi'(\pm T)$. Since the ansatz guarantees that the Kähler form extends positively [H3, HSn, KS1], and so does the Ricci form by assumption, Ψ' is negative at the endpoints.

Taking $\{e_i\}$ to be a simultaneously diagonalizing eigenframe as in the beginning of the section, expressions (2.3) also show that $\kappa_i + \Psi \, \eta_i \geq 0$ on $[-T, T]$, with equality at most at $\pm T$ *exactly* when the corresponding linear factor of Q vanishes at $\pm T$. This follows by Lemma 3.1, and by the positivity of the limiting ratio of the components of the Kähler and Ricci forms in the non-blown-down directions e_i at $\pm T$.

Thus expression (3.3) for the central curvature is positive, and by l'Hôpital's rule,

$$\lim_{\tau \to \pm T} -\frac{R' \circ \Psi(\tau)}{Q(\tau)} > 0. \tag{3.10}$$

Hence the central curvature is positive even on the ends of N.

Since ϱ_M has constant eigenvalues, M is Fano if and only if $\kappa_i > 0$ for all i. As 0 is a value of Ψ, $0 < k_i = k_i + \Psi(\hat{\tau}) n_i$ at some point $\hat{\tau} \in (-T, T)$, so M is indeed Fano.

Conversely, assume $\kappa_i > 0$ for all i, and that

$$C(\tau) = -\Psi'(\tau) \prod_i \frac{\kappa_i + \Psi(\tau) \, \eta_i}{1 - \tau \, \eta_i} > 0 \qquad \text{on } [-T, T], \tag{3.11}$$

with the expression understood at the endpoints in a limiting sense. Since Q is positive in $(-T, T)$, the product of the linear factors in the denominator is positive there. By Proposition 3.2, $\Psi' < 0$ as well, even on $[-T, T]$. Finally, each linear factor in the numerator is nowhere vanishing and positive where Ψ vanishes, hence necessarily positive on $(-T, T)$. Consequently, by expressions (2.3) the Ricci form is positive except perhaps at the ends. But if $\kappa_i + \Psi(\pm T) \, \eta_i = 0$ without $1 \mp T \, \eta_i = 0$, the limiting value (3.10) of C will be zero and not positive. Thus for each direction e_i that is not blown-down $\kappa_i + \Psi(\pm T) \, \eta_i > 0$. Thus the Ricci curvature of g is positive.   □

3.6   Central versus Extremal Metrics

Before demonstrating the existence of central metrics in the examples of the next section, it is shown below that such metrics are not, in any case, extremal.

Proposition 3.8 *Let g be a central Kähler metric arising from the momentum construction. If its central curvature is non-constant, then g is not extremal in the sense of Calabi.*

Proof In this ansatz, the scalar curvature of an *arbitrary* metric depends only on τ, and it was previously noted that the only purely τ-dependent holomorphy potentials are affine. However, if the scalar curvature of a metric is polynomial in τ, then its profile φ is rational in τ; this can be verified from (2.3) by computing the trace of the Ricci endomorphism, see [HSn].

It therefore suffices to show that a central metric of non-constant central curvature is associated to a non-rational profile, and this is clear from the following observations: The polynomials Q and R' have the same degree, so if C is a non-constant affine function, then the polynomials P and R *do not* have the same degree. Consequently, $\Psi = R^{-1} \circ P$ is not a rational function, and therefore the profile φ is not rational either. $\qquad\square$

4 The Case of a One-Dimensional Base

This section is devoted to a concrete and detailed investigation of equations (3.4) and their integrability conditions, in the case where M is a compact Riemann surface of genus p, and $N = \widehat{L} = \mathbf{P}(L \oplus \mathcal{O})$ is a ruled surface.

4.1 The Central Equation and the Integrability Conditions

The ansatz is reviewed below in this special setting. It is enough to consider metrics up to homothety, hence set $T = 1$. The formulas will be written in terms of the Euler number k of M and the Chern number n of L, as in Example 2.3, rather than by means of $\kappa = k/a$ and $\eta = n/a$. Here a parameterizes the base Kähler form via $\omega_M = 2\pi a\, \omega_0$, where ω_0 is indivisible and integral. Since T is fixed, a also parameterizes the Kähler classes on N, under the restriction $a > |n|$. The freedom in the choice of sign for n is eliminated by declaring it non-negative. It is in fact positive, since our interest lies in spaces which may admit metrics of *non-constant* central curvature, and this cannot happen for $n = 0$ (see [Ms2]). The base Ricci form is

$$\rho_M = 2\pi k\, \omega_0 = \frac{k}{a}\omega_M,$$

with $k = 2 - 2p$ the Euler number of the surface. The curvature form of L is

$$\gamma = 2\pi n\, \omega_0 = \frac{n}{a}\omega_M,$$

and the Duistermaat-Heckman measure computes, from the ratio of areas of sections with respect to $\omega_M - \tau\gamma$ and ω_M, to be

$$\left(1 - \tau\frac{n}{a}\right)d\tau = \frac{Q}{a}d\tau.$$

The centrality equation in (3.4) then reads

$$C(\tau) = -\frac{1}{a - n\tau}(k\Psi + \frac{n}{2}\Psi^2)' = b_0 + b_1\tau. \tag{4.1}$$

The function R' of the previous section is $R'(x) = k + nx$, so taking $R(x) = kx + (nx^2)/2 - (k+n/2)$, the convenient antiderivative boundary value $R(d_0) = R(1) = 0$ holds. Thus equation (3.5) reads

$$P(\tau) = -\int_{-1}^{\tau}(b_0 + b_1 x)(a - nx)dx = k\Psi(\tau) + \frac{n}{2}\Psi^2(\tau) - \left(k + \frac{n}{2}\right) = \left(R \circ \Psi\right)(\tau). \tag{4.2}$$

The boundary conditions (2.2) on Ψ are

$$\Psi(\pm 1) = \mp 1.$$

For the linear integrability condition (3.7), one has

$$P(1) = -\left(2b_0 a - 2\frac{b_1 n}{3}\right) = -2k = R(-1),$$

which in turn is used to eliminate b_0:

$$b_0 = \frac{b_1 n}{3a} + \frac{k}{a}. \tag{4.3}$$

Substituting this relation gives a useful form for $P(\tau)$. A short computation gives

$$P(\tau) = -(\tau + 1)\left(k - \frac{kn}{2a}(\tau - 1) + b_1(\tau - 1)\left(-\frac{n}{3}\tau - \left(\frac{n^2}{6a} - \frac{a}{2}\right)\right)\right). \tag{4.4}$$

Turning to the non-linear integrability condition, it is necessary to understand whether R is invertible. In the concrete setting at hand one can compute Ψ directly by completing the square in the right hand side of (4.2):

$$P(\tau) = \frac{n}{2}\left(\left(\Psi + \frac{k}{n}\right)^2(\tau) - \left(\frac{k}{n} + 1\right)^2\right),$$

It follows that the branches of Ψ are of the form

$$\Psi(\tau) = -\frac{k}{n} \pm \sqrt{\frac{2P(\tau)}{n} + \left(\frac{k}{n} + 1\right)^2}. \tag{4.5}$$

In other words, the branches of R^{-1} are $R^{-1}(x) = -k/n \pm \sqrt{(2/n)x + (k/n + 1)^2}$. Recall that R is invertible in $[-1, 1]$ if R' is non-zero in $(-1, 1)$. But $R' = 0$ at $x = -k/n$, so if $|-k/n| \geq 1$, only one of the branches above is used. Otherwise it is necessary to patch both at this interior point. The various possibilities will be listed in detail in the next

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subsection. We conclude by stating the non-linear integrability condition (3.8) (for a single branch):

$$0 = \int_{-1}^{1} (a - n\tau) \left(-\frac{k}{n} \pm \sqrt{\frac{2}{n}P(\tau) + \left(\frac{k}{n} + 1\right)^2} \right) d\tau. \tag{4.6}$$

Possible choices for R^{-1}

The choice of branch of the inverse of R must correspond both to the choice of anti-derivative and to the resulting linear integrability condition:

$$\Psi(-T) = d_0 = R^{-1}P(-T),$$

$$\Psi(T) = -d_\infty = R^{-1}P(T).$$

Since the left hand side is ± 1, and the right hand side computes to $-k/n \pm |k/n \pm 1|$, it follows that the topology of N dictates the allowed branches near each endpoint, according to Table 1:

$Branch:$	$\tau = -1$	$\tau = 1$
$k \leq -n$	$-\sqrt{}$	$-\sqrt{}$
$k < |n|$	$+\sqrt{}$	$-\sqrt{}$
$k \geq n$	$+\sqrt{}$	$+\sqrt{}$

Table 1: Topology and branches of R^{-1} at endpoints.

Note that the second possibility involving a patch of branches cannot occur when $|k| = n$, since k cannot be equal to both n and $-n$ for $n \neq 0$.

The Non-Linear Integrability Condition: Explicit Form, Uniqueness and Existence Considerations

Combining (4.4) with (4.6), the non-linear integrability condition (for one branch) may be written in the form

$$I(b_1) := \int_{-1}^{1} (a - n\tau) \cdot \left(-\frac{k}{n} \pm \sqrt{f_1(\tau)b_1 + f_2(\tau)} \right) d\tau, \tag{4.7}$$

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where

$$f_1(\tau) := \frac{2}{n}(\tau^2 - 1)\Big(\frac{n}{3}\tau + \Big(\frac{n^2}{6a} - \frac{a}{2}\Big)\Big),$$
$$f_2(\tau) := \frac{k}{a}\tau^2 - \frac{2k}{n}\tau - \frac{k}{a} - \frac{2k}{n} + \Big(\frac{k}{n} + 1\Big)^2.$$

For the case of two branches, the positive branch is to be integrated in $[-T, \tau_0]$, and the negative one in $[\tau_0, T]$, with τ_0 as in the previous section.

The coefficient f_1 does not depend on k. Its quadratic factor is non-positive for $-1 \leq \tau \leq 1$, and one can check that for $a > n$, its linear factor is negative in the closed interval. Therefore f_1 is non-negative for $-1 \leq \tau \leq 1$, and vanishes only at the end points.

The free term f_2 is, of course, quadratic, and one can easily check that it is non-negative, decreasing for $k > 0$, increasing for $k < 0$ and equal to the constant 1 when $k = 0$. When $|k/n| = 1$, f_2 factors to:

$$\frac{k}{a}(\tau - 1)(\tau + 1 - 2a/k) \qquad k > 0,$$

$$\frac{k}{a}(\tau + 1)(\tau - 1 + 2a/k) \qquad k < 0.$$

The function $I(b_1)$ is real-analytic on its domain, which is a half-closed infinite interval. On its interior, its derivative equals

$$I'(b_1) = \frac{1}{2}\int_{-1}^{1}(a - n\tau) \cdot \Big((f_1(\tau)b_1 + f_2(\tau))^{-\frac{1}{2}}f_1(\tau)\Big)d\tau$$

which is positive, hence

Proposition 4.1 *There is at most one b_1 in the domain of I for which $I(b_1) = 0$.*

This ensures uniqueness of the metrics. The first step in the existence proof is the evaluation of $I(0)$ in the single branch case:

$$I(0) = \frac{a}{3kn}\Big(-6\,k^2 \pm \frac{1}{n}\Big(|k + n|^3 - |k - n|^3\Big)\Big) = \pm\frac{2na}{3k}.$$

The sign of this expression depends on k. To verify that the non-linear integrability condition is satisfied, it is enough to find a value of b_1 for which the sign of $I(b_1)$ is the reverse of the sign of $I(0)$. Continuity of I will then ensure that the condition is achieved for some other intermediate value of b_1.

The remaining existence considerations are now separated into the different cases, with the following common strategy described first for the one-branch case. The endpoint of the semi-infinite interval consisting the domain of I is a negative number b on which (the monotone function) I achieves its extremum. In all cases b, and then $I(b)$, are either

evaluated or estimated. In some cases the latter is compared with $I(0)$. For the estimation of b one uses various characterizations of it. It is the maximum of $-f_2/f_1$, as well as the unique value of b_1 on which the radicand has a double root at a point $\tau_0 \in [-1, 1]$. Relating τ_0 with b and understanding the former is what allows for the determination of the latter. Moreover, the fact that τ_0 is a *double* root means that the radicand simplifies, and consequently that $I(b)$ can, in principle, be evaluated explicitly. However, determining τ_0 amounts to solving an equation of degree three which yields expressions that are difficult to handle in full generality. We therefore present special cases in which finding τ_0 is easier, or else may be estimated.

The root τ_0 occurs at a point for which $(R' \circ \Psi/Q)(\tau_0) = 0$. In this sense the point considered in the proof of Theorem 3.6 provides an example. Indeed, turning to the two-branch case, τ_0 appears explicitly as a boundary value of both integrals comprising condition (3.9). The latter may be written as $I_+(b_1) + I_-(b_1) = 0$, with I_+ and I_- the monotone functions corresponding to the two branches. The value b may be defined as the (unique) common extremum of both. It also satisfies the characterizations given above and can be related similarly to τ_0. The main conceptual difference from the one-branch case, is that if a central metric exists, then the coefficient b_1 of its central curvature must equal b. This follows from the uniqueness of τ_0 (and of b) as guaranteed in Theorem 3.6, and the requirement that Ψ remains smooth. Thus the interest lies not in the sign of $I_+(b) + I_-(b)$, but in whether or not it vanishes.

In conclusion, we rephrase the geometric significance of b and τ_0 as follows. There is an extremal solution candidate $\Psi = \Psi_b$, which solves the first of equations (3.4). Its associated central curvature vanishes at a point τ_0 which is determined by the topology via $\Psi_b(\tau_0) = -k/n$. It is not a solution to the full system, since it turns out that the non-linear integrability condition forbids the vanishing of the central curvature. However, the claim that there are solutions exactly when $n < |k|$ may be translated into the statement that whenever $\Psi_b(\tau_0)$ lies outside the range $[-d_\infty, d_0]$, Ψ_b can be adjusted to yield a solution with nowhere vanishing central curvature.

4.2   The Second Hirzebruch Surface

A Lower Bound on $I(b_1)$, for $|\mathbf{k}| = \mathbf{n}$

This subsection treats the case $|k| = n$. Since the formula for f_2 changes with the sign of k, the computations will be carried out only for $k > 0$, and then briefly indicated for negative k. Now $k > 0$ means of course $k = 2$, and therefore $n = 2$. Thus the following result is proved for the second Hirzebruch surface:

Theorem 4.2 *The momentum construction yields no central metrics in (any Kähler class of) the second Hirzebruch surface.*

Although the ansatz is used explicitly in the proof, it may turn out to be extraneous in the statement of the result.

Proof The domain of $I(b_1)$ is limited by the requirement that the radicand remain non-negative. This amounts to requiring $b_1 \geq -f_2(\tau)/f_1(\tau)$ in $(-1, 1)$, since $f_1(\tau)$ is positive there. At the endpoints, where f_1 vanishes, $f_2(\tau)$ is non-negative, so the square root is well-defined there for every value of b_1. Hence, to find where I achieves its minimum, it is enough to find the maximum value b of $h(\tau) := f_2(\tau)/(-f_1(\tau))$, which may occur as a limiting value at an endpoint of $[-1, 1]$.

The function h simplifies to

$$h(\tau) = \frac{\frac{2}{a}(\tau - 1)(\tau - (a - 1))}{(1 - \tau^2)(\frac{2}{3}\tau + (\frac{4}{6a} - \frac{a}{2}))} = -\frac{12(\tau - (a - 1))}{(\tau + 1)(4a\tau + 4 - 3a^2)}.$$

A calculation shows that the critical points of h are $3a/2 - 2$ and $a/2$, both larger than 1 since $a > 2$. Hence the best upper bound for h in $(-1, 1)$ is its limiting value at one of the endpoints. One can check that the limit value at $\tau = -1$ is $-\infty$. The maximum b therefore occurs at $\tau = \tau_0 = 1$, and its value is

$$\lim_{\tau \to 1^-} h(\tau) = -\frac{12(2 - a)}{2(4a + 4 - 3a^2)} = -\frac{6}{3a + 2}.$$

Hence $I(b_1)$ is well-defined, so long as $b_1 \geq -6/(3a + 2)$.

Next, since f_2 vanishes at $\tau = 1$ for $k = n = 2$, the radicand factors once there. Substituting $b = -6/(3a + 2)$ for b_1 gives, by the above discussion, a radicand vanishing even to second order at $\tau = 1$. A computation gives:

$$f_1(\tau)b + f_2(\tau) = f_1(\tau)\frac{-6}{3a + 2} + f_2(\tau) = \frac{1}{3a + 2}(\tau - 1)^2(-4\tau + 3a - 2).$$

The integral $I(b)$ is computable. The integrand has a summand containing the radical. This summand simplifies as $\tau - 1$ factors out of the radicand:

$$\int_{-1}^1 (a - 2\tau)\sqrt{\frac{1}{3a + 2}(\tau - 1)^2(-4\tau + 3a - 2)}d\tau =$$

$$\frac{1}{\sqrt{3a + 2}}\int_{-1}^1 (2\tau^2 - (a + 2)\tau + a)\sqrt{-4\tau + 3a - 2}d\tau.$$

Making the substitution

$$u = -4\tau + 3a - 2$$

gives an integral which computes to

$$\frac{1}{112}(3a + 2)^3\left(1 - \left(\frac{3a - 6}{3a + 2}\right)^{\frac{7}{2}}\right) + \frac{1}{20}(2 - a)(3a + 2)^2\left(1 - \left(\frac{3a - 6}{3a + 2}\right)^{\frac{5}{2}}\right)$$

$$+\frac{1}{16}(a - 2)^2(3a + 2)\left(1 - \left(\frac{3a - 6}{3a + 2}\right)^{\frac{3}{2}}\right).$$

This is to be compared with the summand of the integrand in $I(b)$ which does not involve the radical, i.e., with $\int_{-1}^{1}(a - n\tau)(k/n)d\tau = 2ak/n = 2a$. For the borderline value $a = 2$, one gets

$$\frac{8^3}{112} \simeq 4.57 > 4 = 2a.$$

One can similarly check explicitly that this holds also for other values $a > 2$. To proceed in making an estimate valid for all Kähler classes, the term $-\sqrt{(3a - 6)/(3a + 2)} = -\sqrt{1 - (8/(3a + 2))}$ is replaced everywhere by the smaller quantity $-(1 - (4/(3a + 2))) = -(3a - 2)/(3a + 2)$ (using $-\sqrt{1 - x} > -(1 - x/2)$ $x > 0$). Doing this and subtracting $2a$, one sees that $I(b_1)$ is larger than a quantity which computes to

$$\frac{4}{35} \frac{315\,a^4 + 2576\,a^2 + 272 + 420\,a^3 + 112\,a}{(3\,a + 2)^4}.$$

This expression is certainly positive for any $a > 2$. Since $I(b_1)$ is increasing it is therefore everywhere positive, and the non-linear integrability condition is never satisfied. This completes the proof. $\qquad\square$

The Case $-\mathbf{k} = \mathbf{n}$

Similarly, when $k < 0$, and $-k = n$, h has roots a/n and $2 + 3a/n$ both larger than 1, and its maximum in this case will be the limiting value when $\tau = -1$, and equals $-3n/(3a - n)$. With this value the radicand is

$$f_1(\tau)\frac{-3n}{(3a - n)} + f_2(\tau) = -\frac{1}{3a - n}(\tau + 1)^2(2\tau n - 3a - n).$$

The radical-summand of $I(b_1)$ is

$$-1/35\,\frac{3\,a^3 + 11\,a^2 n - 55\,an^2 + 17\,n^3}{n^2} + \frac{3}{35}\,\sqrt{3}\sqrt{\frac{n + a}{3\,a - n}}\left(3\,a^2 n + a^3 + n^3 + 3\,an^2\right)n^{-2},$$

and a similar result is now within reach, at least for specific values of n.

4.3   The First Hirzebruch Surface

Solutions to the Non-Linear Integrability Condition

Proceeding to the constraint $n < |k|$, again the case of positive k, i.e. the first Hirzebruch surface ($n = 1$), is given the more extensive treatment.

Theorem 4.3 *Let $N = \mathbf{P}(L \oplus \mathcal{O})$ be a ruled surface constructed from a line bundle L of Chern number one over a compact Riemann surface of genus zero or two. The momentum construction yields a unique central metric of nowhere-vanishing (non-constant) central*

curvature in every Kähler class of N. The sign of the latter is the same as the sign of the Euler number of the base.

For genus larger than one, the same methods should work for other choices of k and n in the realm $k < -n$.

Proof   Analyzing the radicand, $f_2(\pm 1) = \mp 4 + 5 > 0$. Since both f_2 and f_1 are positive in $(-1, 1)$, $h := -f_2/f_1$ has a limit value equal to $-\infty$ at *both* endpoints. Therefore this function achieves a maximum at a point τ_0 in the interior of $[-1, 1]$. Again denoting by $b = b_1(\tau_0)$ this maximum value gives a radicand vanishing at τ_0: $f_1(\tau_0)b + f_2(\tau_0) = 0$. Note that b is necessarily negative.

Substituting b for b_1 in (4.4) defines $P_b(\tau)$ and then $\Psi_b = R^{-1} \circ P_b$. The notation τ_0 corresponds to the one given in Theorem 3.6 since $[(R' \circ \Psi_b)/Q](\tau_0) = 0$, but one should keep in mind that the function Ψ_b does *not* give rise to a central metric, and in fact its image contains $-k/n = -2$, which is outside the interval $[-d_\infty, d_0]$.

Proceeding to estimate b, one may note as in the previous subsection that τ_0 is a critical point of $h(\tau)$. However, this gives in general an equation of degree four. A simpler relation can be obtained by noticing that τ_0, as a minimum, is also a critical point of $f_1(\tau)b + f_2(\tau) = 2/nP_b(\tau) + (k/n + 1)^2$. Substituting b for b_1 in (4.3) and denoting the result by $b_0(\tau_0)$, τ_0 is a zero of $P_b'(\tau) = -(b_0(\tau_0) + b\tau)(a - n\tau)$. Since $a - n\tau > 0$, one has, using (4.3) again (as well as the non-vanishing of b),

$$\tau_0 = \frac{-b_0(\tau_0)}{b} = -\frac{n}{3a} - \frac{k}{ab} = -\frac{1}{3a} - \frac{2}{ab},$$

or

$$b = -\frac{2}{a\tau_0 + \frac{1}{3}} = -\frac{6}{3a\tau_0 + 1}. \tag{4.8}$$

Here the denominator cannot vanish since in the previous expression $2/(ab) \neq 0$. For making the estimate on b, and then $I(b)$, it will be enough to use $-1 < \tau_0 < 1$. Since b is negative, from the right hand side inequality one deduces

$$b < -\frac{6}{3a + 1}. \tag{4.9}$$

In bounding the summand of $I(b)$ containing the radical, the estimates $f_2(\tau) \leq 5 - 4\tau$, $f_1(\tau) \leq (1/3a)(1 - \tau^2)6a^2 = 2a(1 - \tau^2)$, as well as $\sqrt{1 - x} < 1 - x/2$, $x > 0$, will be employed. The latter is used with $x := -f_1(\tau)b/f_2(\tau) > 0$:

$$\int_{-1}^{1} (a - n\tau)\sqrt{f_1(\tau)b + f_2(\tau)}d\tau =$$

$$\int_{-1}^{1} (a - \tau)\sqrt{f_2(\tau)}\sqrt{1 - \left(-\frac{f_1(\tau)b}{f_2(\tau)}\right)}d\tau \leq$$

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$$\int_{-1}^{1}(a-\tau)\sqrt{f_2(\tau)}\left(1-\left(-\frac{f_1(\tau)b}{2f_2(\tau)}\right)\right)d\tau \;\leq$$

$$\int_{-1}^{1}(a-\tau)\sqrt{f_2(\tau)}d\tau + \frac{b}{2}\int_{-1}^{1}(a-\tau)\frac{2a(1-\tau^2)}{\sqrt{5-4\tau}}d\tau \;< \qquad (4.10)$$

$$\int_{-1}^{1}(a-\tau)\sqrt{f_2(\tau)}d\tau - \frac{3}{3a+1}\int_{-1}^{1}(a-\tau)\frac{2a(1-\tau^2)}{\sqrt{5-4\tau}}d\tau \;=$$

$$\frac{13\,a}{3} - \frac{1}{140}\frac{a\,(-185+154\,a)}{3\,a+1} - \frac{27}{140}\frac{a\,(5+14\,a)}{3\,a+1}.$$

Subtracting $4a = \frac{2ak}{n}$ from this expression gives

$$-\frac{1}{210}\frac{a\,(588\,a-145)}{3\,a+1},$$

which is obviously negative for $a > 1$.

Since $I(b) < 0 < I(0)$, it follows from the previously mentioned continuity argument that there is a value of b_1 in the range

$$b < -\frac{6}{3a+1} < b_1 < 0, \qquad (4.11)$$

for which the right hand side of (4.7) equals $2ak/n$, i.e., a value for which the non-linear integrability condition (4.7) is satisfied, for every Kähler class.

Positivity of the Central Curvature

The above estimate also proves that the solutions found have *positive* central curvature. To see this, first note that if $C(\tau) = 0$ at some point in the interval, then $\tau = -b_1/b_0$ is a critical point of $P(\tau)$, so it must satisfy equation (4.8), and therefore the estimate (4.9), with b_1 replacing b. But since it was shown above that $b_1 > -6/(3a+1)$, the central curvature does not vanish on $[-1,1]$. Its positivity then follows from an earlier remark since the base has positive central curvature. Alternatively, its minimum value is, since $b_1 < 0$, equal to its value at $\tau = 1$, which is $b_0 + b_1 = ((3a+1)b_1+6)/(3a) > 0$, by (4.3) and (4.11). Hence it is positive. By Theorem 3.7, the Ricci tensor of g_φ is positive definite.

The Case of a Base of Genus p > 1

When $k < 0$ and $|k| = -k > n$, then

$$f_1(\tau) = \frac{2}{n}(\tau^2 - 1)\left(\frac{n}{3}\tau + \left(\frac{n^2}{6a} - \frac{a}{2}\right)\right),$$

$$f_2(\tau) = \frac{k}{a}\tau^2 - \frac{2k}{n}\tau - \frac{k}{a} - \frac{2k}{n} + \left(\frac{k}{n} + 1\right)^2,$$

and the latter is increasing in $[-1, 1]$. The equation replacing (4.8) is

$$b = -\frac{k}{a\tau_0 + \frac{n}{3}} = -\frac{3k}{3a\tau_0 + n}.$$

The positivity of the f_i's shows that the sign of the minimum $b_1 := b$ for which $f_1(\tau)b_1 + f_2(\tau) \geq 0$ is still negative (though now I has a maximum at b). Therefore $3a\tau_0 + n < 0$, and one uses the negative bound $-1 < \tau_0$ to get the bound

$$b < -\frac{3k}{n - 3a}.$$

Concentrating on the case $n = 1$, the expression comparable to (4.10) is

$$-\left(\int_{-1}^{1} (a - \tau) \sqrt{f_2(\tau)} d\tau + \frac{b}{2} \int_{-1}^{1} (a - \tau) \frac{2a(1 - \tau^2)}{\sqrt{(k+1)^2 - 2k - 2k\tau}} d\tau + 2ak \right).$$

Notice that $2ak$ was added and not subtracted to accommodate the sign change of the non-linear integrability condition for $k < 0$. For particular values of k this can be evaluated. For example when $k = -2$, one gets

$$\frac{1}{210} \frac{a \left(70 + 798\, a^2 - 135\, a \right)}{-1 + 3\, a},$$

which is positive for $a > 1$. Since this is opposite the sign of $I(0)$, the solution metric thus exists, and has $b_1 > -3k/(1 - 3a)$. The central curvature is nowhere vanishing by the same argument as before. This conclusion and the sign could be found by computing its maximum value. This occurs at $\tau = -1$, and equals $b_0 - b_1 = ((1 - 3a)b_1 + 3k)/(3a) < 0$. Therefore the central curvature is negative. This completes the proof of the theorem. $\square$

4.4 The Genus One Base

The remaining case is $|k| < n$, which involves patching the two branches of Ψ. These meet at a point $\tau_0 \in (-1, 1)$ whose properties are described in Theorem 3.6. Providing an estimate for this case is generally more difficult, since both τ_0 and b appear in the non-linear integrability condition (3.9). This section treats the case $k = 0$, in which τ_0 (and b) can be evaluated explicitly.

Theorem 4.4 *Over a compact manifold M of genus one, the momentum construction yields no central metrics of non-constant central curvature on the space $N = \mathbf{P}(L \oplus \mathcal{O}) \to M$.*

Proof The first simplification occurs since $f_2(\tau) \equiv 1$ identically when $k = 0$. Next, from (4.3),

$$\tau_0 = -\frac{b_0(\tau_0)}{b} = -\frac{n}{3a}.$$

In other words, the vanishing of the Euler number k implies that one has an expression for τ_0 that does not depend on b. On the other hand, since b is also the maximum of $h = -f_2/f_1$, which occurs at τ_0, it evaluates to

$$b = -\frac{1}{f_1(\tau_0)} = -81\frac{na^3}{\left(9\,a^2 - n^2\right)^2}.$$

Therefore one needs to evaluate the integral

$$I_+(b) + I_-(b) = \left(\int_{-1}^{\tau_0} - \int_{\tau_0}^{1}\right)(a - n\tau)\sqrt{1 + f_1(\tau)b}\,d\tau \;\;=$$

$$\left(\int_{-1}^{-\frac{n}{3a}} - \int_{-\frac{n}{3a}}^{1}\right)(a - n\tau)\sqrt{\frac{\left(9\,a^2 - 6\,n\tau\,a + n^2\right)\left(n + 3\,a\tau\right)^2}{\left(9\,a^2 - n^2\right)^2}}d\tau \;\;=$$

$$\left(\int_{-1}^{-\frac{n}{3a}} - \int_{-\frac{n}{3a}}^{1}\right)(a - n\tau)\frac{|n + 3\,a\tau|}{\left(9\,a^2 - n^2\right)}\sqrt{\left(9\,a^2 - 6\,n\tau\,a + n^2\right)}d\tau,$$

Just as in the case of the second Hirzebruch surface, this integral can be computed, and the result is

$$-\frac{2}{105}\frac{n\left(n^4 - 42\,a^2 n^2 + 105\,a^4\right)}{\left(n - 3\,a\right)a^2\left(n + 3\,a\right)}.$$

The bracketed term of the numerator is larger than $n^4 - 42a^4 + 105a^4 = n^4 + 63a^4$ and so is never zero. Thus $I_+(b) + I_-(b) \neq 0$, and by the general argument for the 2-branch situation, the non-linear integrability condition is never satisfied. $\square$

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