SPECIAL KÄHLER-RICCI POTENTIALS AND RICCI SOLITONS

GIDEON MASCHLER

Abstract. On a manifold of dimension at least six, let (g, τ) be a pair consisting of a Kähler metric g which is locally Kähler irreducible, and a nonconstant smooth function τ. Off the zero set of τ, if the metric $\widehat{g} = g/\tau^2$ is a gradient Ricci soliton which has soliton function $1/\tau$, we show that $\widehat{g}$ is Kähler with respect to another complex structure, and locally of a type first described by Koiso, and also Cao. Moreover, τ is a special Kähler-Ricci potential, a notion defined in earlier works of Derdzinski and Maschler. The result extends to dimension four with additional assumptions. We also discuss a *Ricci-Hessian equation*, which is a generalization of the soliton equation, and observe that the set of pairs (g, τ) satisfying a Ricci-Hessian equation is invariant, in a suitable sense, under the map $(g, \tau) \to (\widehat{g}, 1/\tau)$.

1. Introduction

In this paper we study pairs (g, τ) on a manifold M of dimension larger than two, where g is a Riemannian metric and τ is a smooth nonconstant function. In this context, an important role will be played by the map $(g, \tau) \to (\widehat{g}, \widehat{\tau}) = (g/\tau^2, 1/\tau)$, considered as an involution on the set of all pairs satisfying obvious restrictions (see §2.2). We will call $(\widehat{g}, \widehat{\tau})$ the *associated pair* of (g, τ).

We say that a pair satisfies a *Ricci-Hessian equation* if

$$\alpha \nabla d\tau + \mathrm{r} = \gamma\, g \tag{1.1}$$

holds for the Hessian of τ, the Ricci tensor r of g, and some C^∞ coefficient functions α and γ. If α and γ are constant, the pair, or sometimes just the metric, is called a gradient Ricci soliton, and if α is additionally nonzero, τ is called the soliton function.

Before stating our main result, we note a closely related fact: the set of pairs satisfying a Ricci-Hessian equation is invariant under our involution. The latter is well-defined once the domain of allowed pairs is further restricted (see §2.2). In this setting we call the involution the duality map.

A part of our main result may be stated informally as follows. Consider two subsets of the set of all pairs on M: those for which the metric is Kähler (and locally irreducible in a suitable sense, see below), and pairs which are gradient Ricci solitons. Assuming a restriction on the dimension of the manifold, if the involution maps an element of the first subset to an element of the second one, the latter element lies in the intersection of the two subsets. More precisely,

Theorem A. *Let M be a manifold of dimension at least six, and (g, τ) a pair as above, with g a Kähler metric. Suppose g is not a local product of Kähler metrics in any neighborhood of some point of M. If the associated pair $(\widehat{g}, \widehat{\tau})$ is a gradient Ricci soliton, then, on $M \setminus \tau^{-1}(0)$, the metric $\widehat{g}$ is Kähler.*

The complex structures giving the Kähler structures of g and $\widehat{g}$ are oppositely oriented. Also, with extra assumptions (see paragraph following Remark 3.2), the result extends to real dimension four.

The proof, in fact, yields much more information on both pairs. First, τ is a special Kähler-Ricci potential. This notion (Definition 3.4) was first defined in [6], and implies that τ is a Killing potential, and that (1.1) holds in some (generically nonempty) open subset of the manifold. Second, the Kähler-Ricci soliton $(\widehat{g}, \widehat{\tau})$ is locally of a type first described by Koiso in [10] (see also [4]).

Although our result is of a local character, one should note that there exist compact manifolds, specifically toric Fano ones, which admit Kähler-Ricci solitons [12], most of which are not of the form found by Koiso.

In the following we describe a few related problems of a broader scope. The involution above is defined in part via a conformal change, and one can ask whether, starting with a Kähler metric g, one can find a metric g/τ^2, for some function τ as above, which is a Ricci soliton. The case where g/τ^2 is Einstein, was the subject of the study of [6, 7, 8], where local and global classifications were given, and, in all even dimensions larger than four, τ turns out to be, in fact, a special Kähler-Ricci potential. In dimension four this need not be the case, and even for compact manifolds, a counterexample was recently shown to exist in [5].

For the case of Ricci solitons, a similar general classification is not known, even if one *assumes* that τ is a special Kähler-Ricci potential. Our result can be considered a first step in an attempt to answer this question, for the case where the resulting Ricci soliton has a very special soliton function. In fact, an alternative formulation of Theorem A is possible, which trades an assumption on τ with a more general allowed soliton function: one assumes that τ is a Killing potential, and then the conclusion follows not just for the associated pair, but for any pair $(\widehat{g}, f)$, where f is a (nonconstant smooth) function locally dependent only on τ. This observation follows using Proposition 3.1 below, which also serves to motivate the expression defining $\widehat{\tau}$.

In various talks, G. Tian has asked whether there exist compact non-Kähler Ricci solitons in dimension four. Extending the question to all dimensions, one may answer it affirmatively via constructions involving products. Ignoring these fairly simple examples, one can try to produce such a Ricci soliton in the conformal class of a Kähler metric (at least in dimension four, it is not too difficult to see that there can be no more than two Kähler metrics in a given conformal class). Our result can be regarded as implying that, in a special case, such an attempt will fail.

Finally, recall the result of Schur, stating that if $\mathrm{r} = \phi\, g$ for some function ϕ, then, except in dimension two, ϕ must be constant. A similar principle holds for Kähler-Ricci solitons, for the coefficient of g, and one may ask whether it holds for any Ricci soliton, or, equivalently, whether there exist pairs for which (1.1) holds with the coefficient of $\nabla d\tau$ constant, but not the coefficient of g. In unpublished work which may be regarded as a global extension of this paper, A. Derdzinski has shown that such pairs do exist on compact manifolds. In fact, on these manifolds there are families of pairs (g, τ), with g Kähler, for which the associated pairs $(\widehat{g}, \widehat{\tau})$ are each of this type, and are obtained by deforming one of the Einstein metrics in [8].

This paper is arranged as follows. Riemannian preliminaries on duality and Ricci solitons appear in §2. Consequences of the Kähler condition for (1.1), along with a review of the basic properties of metrics with a special Kähler-Ricci potential, are given in §3. Ordinary differential equations associated with a large class of such metrics are studied in §4, especially in relation to the assumption that the associated pair forms a Ricci soliton, and we give an analysis of their solution set. After recalling the geometric structure of a Kähler metric admitting a special Kähler-Ricci potential, and presenting a duality result in this context in §5, we summarize our results in §6 by proving Theorem 6.1, from which Theorem A easily follows. Our conventions throughout closely follow [6].

2. Ricci-Hessian equations, duality and Ricci solitons

2.1. **Conformal changes.** Let (M, g) be a Riemannian manifold of dimension n, and $\tau : M \to \mathbb{R}$ a nonconstant C^∞ function. We write metrics conformally related to g in the form $\widehat{g} = g/\tau^2$, and set $Q = g(\nabla\tau, \nabla\tau)$. The metric $\widehat{g}$ will always be considered on its domain of definition, i.e. the set $M \smallsetminus \tau^{-1}(0)$. With respect to $\widehat{g}$, the Hessian of a given C^2 function f on $M \smallsetminus \tau^{-1}(0)$ is given by

$$\widehat{\nabla} df \;=\; \nabla df \;+\; \tau^{-1}[2\, d\tau \odot df \;-\; g(\nabla\tau, \nabla f) g], \tag{2.1}$$

where $d\tau \odot df = (d\tau \otimes df + df \otimes d\tau)/2$. We will be concerned primarily with the case where $df \wedge d\tau = 0$, i.e., at points where $d\tau \neq 0$, f is given locally as a composition $f = H \circ \tau$. In this case, (2.1) becomes $\widehat{\nabla} df \;=\; f' \,\nabla d\tau + (f'' + 2\tau^{-1} f')\; d\tau \otimes d\tau - f' \,\tau^{-1}\, Q\, g$, with $'$ denoting differentiation with respect to τ. For the particular choice of $f = \tau^{-1}$, this expression simplifies:

$$\widehat{\nabla} d\tau^{-1} \;=\; -\tau^{-2}(\,\nabla d\tau \;-\; \tau^{-1}\, Q\, g), \qquad \text{if } \widehat{g} = g/\tau^2. \tag{2.2}$$

Finally, recall the conformal change expression relating the Ricci tensors of g and $\widehat{g}$, with Δ denoting the Laplace operator:

$$\widehat{\mathrm{r}} \;=\; \mathrm{r} \;+\; (n-2)\,\tau^{-1}\nabla d\tau \;+\; \big[\tau^{-1}\Delta\tau \;-\; (n-1)\,\tau^{-2} Q\big]\, g. \tag{2.3}$$

2.2. **Ricci-Hessian equations and duality.** With M, g, τ and other notations as above, we say that the pair (g, τ) satisfies a *Ricci-Hessian equation* on M (or often just on an open set of M), if (1.1) holds there. We record this equation more explicitly as

$$\alpha\nabla d\tau + \mathrm{r} = \gamma\, g, \quad \text{with } \tau \text{ nonconstant}, \tag{2.4}$$

where $\nabla d\tau$ and r are as above, and α, γ are C^∞ coefficient functions. What we will call duality may be regarded informally as an involution on the space of pairs satisfying (2.4):

Proposition 2.1 (Derdzinski)**.** *Let M have dimension $n > 3$, and suppose a pair (g, τ) as above satisfies a Ricci-Hessian equation (2.4) on M. Then the pair $(\widehat{g}, \widehat{\tau}) = (g/\tau^2, 1/\tau)$ also satisfies a Ricci-Hessian equation $\widehat{\alpha}\widehat{\nabla} d\tau + \widehat{\mathrm{r}} = \widehat{\gamma}\, \widehat{g}$, on $M \smallsetminus \tau^{-1}(0)$, with coefficients*

$$\widehat{\alpha} = (n-2)\tau - \tau^2\alpha, \quad \widehat{\gamma} = \gamma\tau^2 - (1 + \alpha\tau)Q + \tau\Delta\tau. \tag{2.5}$$

In fact, letting β denote the coefficient of g in (2.3), one has, by (2.2) and (2.3),

$$\begin{aligned}\widehat{\alpha}\,\widehat{\nabla}d\widehat{\tau} + \widehat{\mathrm{r}} &= ((n-2)\tau - \tau^2\alpha)(-\tau^{-2}\,\nabla d\tau\, + \tau^{-3}Qg) + \mathrm{r}\, + (n-2)\tau^{-1}\nabla d\tau + \beta g\\ &= \alpha\,\nabla d\tau\, + \mathrm{r}\, + (\widehat{\alpha}\tau^{-3}Q + \beta)g = (\gamma + \widehat{\alpha}\tau^{-3}Q + \beta)\tau^2\widehat{g},\end{aligned}$$

and one easily checks that the last expression is $\widehat{\gamma}\,\widehat{g}$.

Remark 2.2. As mentioned in the Introduction, the pair $(\widehat{g}, \widehat{\tau})$ will be called the associated pair. It is not necessarily defined on all of M. Hence, to consider $(g, \tau) \to (\widehat{g}, \widehat{\tau})$ as an involution, one must regard it as having a domain consisting, for example, of pairs which are defined on *some* open set of M. Furthermore, in order to understand Proposition 2.1 as indicated in the Introduction, i.e. as a statement about an invariant set for this involution, one must further require that the pairs are defined on some open subset that does not contain points where $\nabla d\tau$ is a multiple of g. This is necessary, since otherwise, a pair (g, τ) may not determine the coefficients α, γ uniquely at every point of M. Finally, further such restrictions may result from considering pairs satisfying (2.4) for coefficients α and γ which have, say, isolated singularities.

To verify the involutive property, one easily checks that $\widehat{\widehat{\alpha}} = \alpha$, while $\widehat{\widehat{\gamma}} = \gamma$ follows from this, as g, τ and α determine γ uniquely. One can also check the last relation directly, using the following formulas for the two functions $\widehat{Q} = \widehat{g}(\widehat{\nabla}\widehat{\tau}, \widehat{\nabla}\widehat{\tau})$ and $\widehat{\Delta}\widehat{\tau}$:

$$\widehat{Q} = \tau^{-2}Q, \qquad \widehat{\Delta}\widehat{\tau} = n\tau^{-1}Q - \Delta\tau.$$

Remark 2.3. For any pair (g, τ) satisfying (2.4), one can produce another such pair by a (nonconstant) affine change in τ. If this affine change involves only a change by an additive constant, it leaves equation (2.4) invariant. This freedom induces, of course, a freedom in the choice of $\widehat{\tau}$, which will be exploited in Proposition 5.1.

2.3. **Ricci solitons.** A *Ricci soliton* [9] is a Riemannian manifold $(M, \widehat{g})$ such that $\pounds_v\widehat{g} + \widehat{\mathrm{r}} = e\,\widehat{g}$ for some constant e and C^∞ vector field v on M. Here $\pounds_v$ is the Lie derivative and $\widehat{\mathrm{r}}$ denotes the Ricci tensor of $\widehat{g}$. We will only be interested in *gradient* Ricci solitons, in which M admits a C^∞ function $f : M \to \mathbb{R}$ with

$$\widehat{\nabla}df\, + \widehat{\mathrm{r}} = e\,\widehat{g} \quad \text{for a constant } e. \tag{2.6}$$

We will call f the *soliton function.* By a result of Perelman [11, Remark 3.2], every compact Ricci soliton $(M, \widehat{g})$ is a gradient Ricci soliton. Recall also that a metric is Einstein if its Ricci tensor is a multiple of it.

Thus a gradient Ricci soliton is, alternatively, a pair $(\widehat{g}, f)$ satisfying a Ricci-Hessian equation with constant coefficients. Using (2.2) and (2.3), or, more naturally, the duality of Proposition 2.1 (slightly modified to allow $\widehat{\tau}$ to be multiplied by a constant), we have

Proposition 2.4. *Let (M, g) be a Riemannian manifold of dimension $n > 2$ and τ a nonconstant C^∞ function. The Ricci soliton equation $\widehat{\nabla}d\,(b\tau^{-1}) + \widehat{\mathrm{r}} = e\,\widehat{g}$, with b a constant, holds for $\widehat{g} = \tau^{-2}g$ on $M \smallsetminus \tau^{-1}(0)$, if and only if g satisfies a Ricci-Hessian equation (2.4) with coefficients*

$$\alpha = (n-2)\,\tau^{-1} - b\,\tau^{-2}, \quad \gamma = e\,\tau^{-2} - \tau^{-1}\Delta\tau + \big((n-1)\,\tau^{-2} - b\,\tau^{-3}\big)\,Q. \tag{2.7}$$

Remark 2.5. The introduction of the constant b serves to compare with the conformally Einstein case, which occurs when $b = 0$: relations (2.7) with $b = 0$ are implied by [6, (6.1) and (6.2)], which hold in that case. It follows from this that an Einstein metric cannot *also* satisfy a Ricci soliton equation (2.6) with the soliton function a *nonzero* multiple of τ^{-1}. Note that for other nonconstant soliton functions, this is possible. Finally, the constant b also plays a role in the proof of the alternative version, mentioned in the Introduction, of Theorem A.

Remark 2.6. Here, in Proposition 3.1 and in Remark 3.2 we briefly consider conformal changes of g, which yield a gradient Ricci soliton $\widehat{g}$ having a more general soliton function f. First, if f is an arbitrary smooth function, applying equations (2.1) and (2.3) yields

$$\begin{aligned} \mathrm{r} \;+\; & (n-2)\tau^{-1}\nabla d\tau + \nabla df + 2\tau^{-1}\, d\tau \odot df \\ = \; & \big[e\tau^{-2} + (n-1)\tau^{-2}Q - \tau^{-1}\Delta\tau + \tau^{-1}g(\nabla\tau, \nabla f)\big]\, g. \end{aligned} \tag{2.8}$$

If now $df \wedge d\tau = 0$, this gives

$$\begin{aligned} \mathrm{r} \;+\; & \big(f' + (n-2)\,\tau^{-1}\big)\,\nabla d\tau + \big(f'' + 2\tau^{-1}f'\big)\, d\tau \otimes d\tau \\ = \; & \big[e\,\tau^{-2} - \tau^{-1}\Delta\tau + \big((n-1)\tau^{-2} + \tau^{-1}f'\big)\, Q\big]\, g. \end{aligned} \tag{2.9}$$

As an aside we note that a particular choice for f will eliminate the Hessian term. Namely, setting $f = -(n-2)\log|\tau|$, the metric $\widehat{g}$ is a Ricci soliton metric precisely when

$$\mathrm{r} - (n-2)\,\tau^{-2}\, d\tau \otimes d\tau = \big[\tau^{-2}\,(e+Q) - \tau^{-1}\Delta\tau\big]\, g\,.$$

However, this simple equation implies that g cannot be Kähler (unless $n = 2$ or τ is constant). This is one other reason, apart from duality considerations, and Proposition 3.1 below, why we will focus on the case of a soliton function proportional to τ^{-1}.

3. Ricci-Hessian equations and special Kähler-Ricci potentials

3.1. **The Ricci-Hessian equation and Ricci solitons.** Let (M, J) be a complex manifold, with J the associated almost complex structure. Suppose g is a Kähler metric on M, i.e a Riemannian metric for which J is parallel. Let (g, τ) be a pair satisfying the Ricci-Hessian equation (2.4) on M. The Kähler property implies that both g and r are Hermitian, hence so is $\nabla d\tau$ on the support of α. Often in applications, this support will be a dense set in M. This property of $\nabla d\tau$ is equivalent to the statement that τ is a Killing potential, i.e. a C^∞ function for which $J\nabla\tau$ is a Killing vector field (cf. [6, Lemma 5.2]).

In the Kähler case, if g/τ^2 is a Ricci soliton, certain restrictions on the soliton function force it to be proportional to, or at least affine in τ^{-1}.

Proposition 3.1. *Let (M, g) be a Kähler manifold with a Killing potential τ, and $\widehat{g} = g/\tau^2$ a Ricci soliton with a smooth τ-dependent soliton function f. Then f is an affine function in τ^{-1}.*

Proof. As (2.9) holds under our assumptions, and $d\tau \otimes d\tau$ is the only term in it that is not Hermitian, its coefficient $f'' + 2\tau^{-1}f'$ must vanish, implying the conclusion. □

Remark 3.2. If the Killing assumption above is replaced by (2.4) for some τ, α not identically zero and γ, the conclusion still follows on the support of α. If one then drops the τ-dependence assumption on the soliton function f, all that (2.4) implies, in combination with (2.8), is that $\nabla df + 2\tau^{-1} d\tau \odot df = (2\tau^2)^{-1} \pounds_{(\tau^2 \nabla f)}\, g$ is Hermitian on the support of α.

We will be especially interested in the case where α and γ in (2.4) are functions of τ. We note that, this always holds for α in (2.7), while it holds for γ there if both $d\tau \wedge d\Delta\tau = 0$ and $d\tau \wedge dQ = 0$. One may attempt to weaken these assumptions using methods akin to those of [6, (6.5) and Proposition 6.4]. We choose to follow here the quicker approach of [1, §1.4], which, however, works only for $m > 2$.

Proposition 3.3. *If (2.4) holds for a Kähler metric of complex dimension $m > 2$, and $d\alpha \wedge d\tau = 0$, then $d\gamma \wedge d\tau = 0$.*

Proof. Composing (2.4) with J and applying d to the result gives $d\alpha \wedge d(\imath_{\nabla\tau}\omega/2) = (d\alpha/d\tau)\, d\tau \wedge d(\imath_{\nabla\tau}\omega/2) = d\gamma \wedge \omega$, using [6, (5.3)] (here ω is the Kähler form of g). Exterior multiplication with $d\tau$ gives $d\tau \wedge d\gamma \wedge \omega = 0$, and the result follows because the operation $\wedge\omega$ is injective on 2-forms for $m > 2$. ☐

Thus, the coefficients (2.7) of the Ricci-Hessian equation will be functions of τ, provided (M, g, τ) is Kähler of dimension $m > 2$, and g/τ^2 is a Ricci soliton with soliton function proportional to τ^{-1}.

3.2. **Special Kähler-Ricci potentials.** Below, we denote by M_τ the complement, in a manifold M, of the critical set of a smooth function τ. For a Killing potential on a Kähler manifold, M_τ is open and dense in M.

Definition 3.4. [6] A nonconstant Killing potential τ on a Kähler manifold (M, J, g) is called a *special Kähler-Ricci potential* if, on the set M_τ, all non-zero tangent vectors orthogonal to $\nabla\tau$ and $J\nabla\tau$ are eigenvectors of both $\nabla d\tau$ and r.

We will call a metric admitting a special Kähler-Ricci potential an *SKR metric*, and occasionally will declare (g, τ) to be an *SKR pair*. Among the more important characteristics of such a metric is the existence of a Ricci-Hessian equation. More precisely

Proposition 3.5. [6, Corollary 9.2, Remarks 7.1 and 7.4] *Let (M, g) be a Kähler manifold of complex dimension $m \geq 2$. If (2.4) holds for some C^∞ functions α, γ and (nonconstant) τ, with $d\alpha \wedge d\tau = 0$, $d\gamma \wedge d\tau = 0$ and $\alpha\, d\alpha \neq 0$ everywhere in M_τ, then τ is a special Kähler-Ricci potential. Conversely, if (M, g) admits a special Kähler-Ricci potential τ, then (2.4) holds on an open subset of M_τ, namely away from points where $\nabla d\tau$ is a multiple of g.*

Remark 3.6. In [6], we have actually written the Ricci-Hessian equation in the form $\nabla d\tau + \chi \mathrm{r} = \sigma\, g$. Note that the domains of the coefficient functions may vary as one switches between these two forms. In general, any statement involving the Ricci-Hessian equation of an SKR metric refers to the largest domain on which (2.4) holds. Moreover, this change results in a slightly different statement of the first part of Proposition 3.5, while to get the second part (and its proof), one need only to switch r with $\nabla d\tau$ in [6, second paragraph of Remark 7.4].

Corollary 3.7. If (M, g) is Kähler, of complex dimension $m > 2$, and $\widehat{g} = g/\tau^2$ is a Ricci soliton, with soliton function $b\tau^{-1}$, where b is a constant, then τ is a special Kähler-Ricci potential.

Proof. We combine Propositions 2.4, 3.3 and 3.5, except that for α in (2.7), $\alpha\, d\alpha = 0$ on the set where $\tau = (n-2)/b$ and on the set where $\tau = 2b/(n-2)$, so that τ is a special Kähler-Ricci potential, and hence a Killing potential away from these sets. By [6, Lemma 5.2], $\nabla d\tau$ is Hermitian away from these sets, yet it is also clearly Hermitian in the interior of each of these sets, so that by continuity, it is Hermitian on all of M_τ. Again using [6, Lemma 5.2], this means that $\nabla\tau$ is holomorphic on M_τ, which implies that the interiors of the above mentioned two sets are empty. As the SKR condition is defined by equalities, continuity now implies that τ satisfies it on all of M_τ. □

By [6, Definition 7.2, Remark 7.3], the SKR condition on (g,τ) is equivalent to the existence, on M_τ, of an orthogonal decomposition $TM = \mathcal{V} \oplus \mathcal{H}$, with $\mathcal{V} = \mathrm{span}(\nabla\tau, J\nabla\tau)$, along with four smooth functions ϕ, ψ, λ, μ which are pointwise eigenvalues for either $\nabla d\tau$ or r, i.e., they satisfy

$$\begin{aligned} \nabla d\tau|_{\mathcal{H}} &= \phi\, g|_{\mathcal{H}}, & \nabla d\tau|_{\mathcal{V}} &= \psi\, g|_{\mathcal{V}}, \\ \mathrm{r}|_{\mathcal{H}} &= \lambda\, g|_{\mathcal{H}}, & \mathrm{r}|_{\mathcal{V}} &= \mu\, g|_{\mathcal{V}}. \end{aligned} \tag{3.1}$$

This decomposition is also r- and $\nabla d\tau$-orthogonal.

Remark 3.8. By [6, Lemma 12.5], ϕ either vanishes identically on M_τ, or never vanishes there. In the former case, g is reducible to a local product of Kähler metrics near any point (see [6, Corollary 13.2] and [7, Remark 16.4]). In the latter case, we call g a *nontrivial* SKR metric.

Remark 3.9. For a nontrivial SKR metric, consider $c = \tau - Q/(2\phi)$, with $Q = g(\nabla\tau, \nabla\tau)$, and $\kappa = \mathrm{sgn}(\phi)(\Delta\tau + \lambda\, Q/\phi)$, regarded as functions $M_\tau \to \mathbb{R}$. By [6, Lemma 12.5], c is constant on M_τ, and will be called the *SKR constant.* In any complex dimension $m \geq 2$, we will call a nontrivial SKR metric *standard* if κ is constant (and also use "standard SKR pair" as a designation for (g,τ)). According to [6, §27, using (10.1) and Lemma 11.1], κ is in fact constant if $m > 2$, so that the designation "standard" involves an extra assumption as compared with "nontrivial" only when $m = 2$. The geometric meaning of κ will be recalled in §5.1.

Remark 3.10. Using Proposition 3.5, for any SKR metric, we see that (2.4) holds at points of M_τ for which $\phi \neq \psi$. On this subset, we regard (2.4) as an equality of operators, and equate eigenvalues to obtain $\alpha\phi + \lambda = \gamma = \alpha\psi + \mu$, so that

$$\lambda - \mu = (\psi - \phi)\alpha. \tag{3.2}$$

According to [6, Lemma 11.1a], Q, $\Delta\tau$, ϕ, ψ and μ are locally C^∞ functions of τ on M_τ. If g is a standard SKR metric, λ is also such a function, as one concludes from the equation defining κ. Hence, by (3.2), the same holds for α on its domain of definition.

4. Associated differential equations

4.1. **The SKR differential equation.** A number of ordinary differential equations are associated with nontrivial SKR metrics. Special cases of these were given in [6]. They are derived below from the Ricci-Hessian equation (2.4), i.e.

$$\alpha\nabla d\tau + \mathrm{r} = \gamma\, g.$$

In the next proposition, α will be as in (2.4), ϕ as in (3.1), c and κ as in Remark 3.9 and a prime denotes the derivative operator $d/d\tau$.

Proposition 4.1. *Let (g, τ) be an SKR pair with g nontrivial, on a manifold M of complex dimension m. Then, the equation*

$$(\tau - c)^2\phi'' + (\tau - c)[m - (\tau - c)\alpha]\phi' - m\phi = -\mathrm{sgn}(\phi)\kappa/2. \tag{4.1}$$

holds at points of M_τ for which $\phi'(\tau)$ is nonzero. If g is standard, (4.1) is an ordinary differential equation, which, upon differentiation and division by $\tau - c$, takes the homogeneous form

$$(\tau - c)\phi''' = [(\tau - c)\alpha - m - 2]\,\phi'' + [(\tau - c)\alpha' + 2\alpha]\,\phi'. \tag{4.2}$$

A special case of equation (4.2) was important in [8], but is given here mainly for the sake of completeness. We will only be using equation (4.1).

Proof. By Remark 3.10, on M_τ, each of Q, ϕ, ψ, $\Delta\tau$ and μ is locally a function of τ. In fact, we have

$$\begin{array}{llll} \text{a)} & \psi = \phi + (\tau - c)\phi', & \text{b)} & \psi' = 2\phi' + (\tau - c)\phi'', \\ \text{c)} & \Delta\tau = 2m\phi + 2(\tau - c)\phi', & \text{d)} & \mu = -(m+1)\phi' - (\tau - c)\phi''. \end{array} \tag{4.3}$$

Namely, [6, Lemma 11.1(b)] gives $2\psi = Q'$, which yields (4.3.a) (and hence (4.3.b)), since $Q = 2(\tau - c)\phi$ due to the definition of c. Next, (4.3.c) is immediate from (4.3.a), as $\Delta\tau = \mathrm{tr}_g\nabla d\tau = 2\psi + 2(m - 1)\phi$. Finally, $2\mu = -(\Delta\tau)'$ by [6, Lemma 11.1(b)], and so, differentiating (4.3.c), we obtain (4.3.d).

Next, by Remark 3.10, equation (2.4) holds at points of M_τ for which $\phi \neq \psi$. Since, in view of (4.3.a), the latter inequality holds when $(\tau - c)\phi' \neq 0$, and $\tau \neq c$ on M_τ (as $Q = 2(\tau - c)\phi$ and $Q > 0$ on M_τ), we see that this set consists exactly of the points of M_τ for which $\phi'(\tau)$ is nonzero.

As on this subset of M_τ, (2.4) holds, so does (3.2), which along with $Q = 2(\tau - c)\phi$ and the definitions of κ and c easily yields $\mathrm{sgn}(\phi)\kappa/2 = \Delta\tau/2 + (\tau - c)\lambda = \Delta\tau/2 + (\tau - c)[\mu + (\psi - \phi)\alpha]$. Replacing μ, ψ and $\Delta\tau$ with the expressions provided by (4.3), we get (4.1). If g is standard, κ is constant, so α is a function of τ by Remark 3.10. Hence equation (4.1) is an ordinary differential equation, and (4.2) then follows as described in the body of the proposition. □

Remark 4.2. A converse statement to this result can be made, where (4.1) implies (2.4) for a standard SKR metric, under the following extra assumptions.

Let ϕ be *globally* a function of τ, in the sense that it is the composite of τ with some C^∞ function $I' \to \mathbb{R}$ on the image interval $I' = \tau(M_\tau)$. (That I' is indeed an interval is known, see [7, §10 and §11].) Assuming ϕ', as a function of τ, is nonzero at all points of a dense subset of I', and (4.1) holds on I' for a C^∞ function $\alpha : I' \to \mathbb{R}$, we obtain that (2.4) is satisfied on M_τ by $\alpha = \alpha(\tau)$ and some γ.

In fact, the assumption involving ϕ' means, as we have seen in the proof above, that (2.4) holds on a dense subset of M_τ, with some α that must coincide with the one above: they both satisfy (4.1) with the same ϕ on a dense subset of I', and hence everywhere in I'.

4.2. **The differential equations in relation to Ricci solitons.** Let (g,τ) be a standard SKR pair on a manifold M of complex dimension m. Suppose g/τ^2 is a Ricci soliton with soliton function $b\tau^{-1}$, where b is a constant. By Proposition 2.4, equation (2.4) holds on $M \smallsetminus \tau^{-1}(0)$, with

$$\alpha = (2(m-1)\,\tau - b)/\tau^2. \tag{4.4}$$

Hence, in this case, the differential equations appearing in Proposition 4.1 take the form

$$\tau^2(\tau-c)^2\phi'' + (\tau-c)\left[m\tau^2 - (\tau-c)(2(m-1)\tau-b)\right]\phi' - m\tau^2\phi = -\mathrm{sgn}\,(\phi)\kappa\,\tau^2/2 \tag{4.5}$$

and

$$\tau^3(\tau-c)\phi''' = [(m-4)\tau^3 - (2(m-1)c+b)\tau^2 + bc\tau]\phi'' + [2(m-1)\tau(\tau+c) - 2bc]\phi'.$$

These equations hold for τ values corresponding to points of M_τ on which $\phi'(\tau)$ is nonzero.

Another ordinary differential equation is obtained on the same set as follows. The term γ in (2.4) is given, by Remark 3.10, as $\gamma = \alpha\psi + \mu$, and we substitute for ψ and μ their respective expressions (4.3.a) and (4.3.d), to obtain

$$\gamma = \alpha\phi + (\alpha(\tau-c) - (m+1))\,\phi' - (\tau-c)\phi''.$$

In the case at hand, γ also has an expression derived from the last term of (2.7), in which we replace Q by $2(\tau-c)\phi$, and $\Delta\tau$ by (4.3.c). Equating the two expressions, and replacing α by (4.4), we get after rearranging terms and multiplying by τ that

$$\begin{aligned} -\tau^3(\tau-c)\phi'' \;&+\; [(2m\tau-b)\tau(\tau-c) - \tau^3(m+1)]\phi' \\ &+\; [2(2m-1)\tau^2 - b\tau + 2(\tau-c)(b-(2m-1)\tau)]\phi = e\tau. \end{aligned} \tag{4.6}$$

The fact shown below, that (4.6) is not, in general, a consequence of (4.5), is the main local difference between the case where the SKR metric is conformal to a non-Einstein Ricci soliton of the type we are considering, with $b \neq 0$, and the one where it is conformal to an Einstein metric ($b = 0$). The latter was the object of study of [6, 7, 8].

4.3. **Solutions of the system (4.5)–(4.6).** To examine the solutions of the system (4.5)–(4.6), we note the following

Lemma 4.3. *Let $\{\phi' + p\phi = q,\ A\phi'' + B\phi' + C\phi = D\}$ be a system of ordinary differential equations in the variable τ, with coefficients p, q, A, B, C and D that are rational functions. Then, on any nonempty interval admitting a solution ϕ, either*

$$A(p^2 - p') - Bp + C = 0 \tag{4.7}$$

holds identically, or

$$\phi = (D - A(q' - pq) - Bq)\,/\,\big(A(p^2-p') - Bp + C\big)\,. \tag{4.8}$$

holds away from the (isolated) singularities of the right hand side.

Proof. Let ϕ be a solution on an interval as above. We have $\phi' = q - p\phi$, so that $\phi'' = q' - p'\phi - p\phi' = q' - p'\phi - p(q-p\phi) = (p^2-p')\phi + q' - pq$. Substituting this in the second equation, while collecting terms involving ϕ, gives

$$\big(A(p^2-p') - Bp + C\big)\,\phi + A(q' - pq) + Bq = D,$$

from which the result follows at once. □

To apply this lemma to (4.5)–(4.6), we multiply (4.6) by $(\tau - c)/\tau$ and add it to (4.5), getting a first order equation which, after multiplying by τ, simplifies to

$$\begin{aligned} \tau^2(\tau-c)(\tau-2c)\phi' \;&+\; \big[-m\tau^3 + (2(2m-1)c+b)\tau^2 - (2(2m-1)c^2+3bc)\tau + 2bc^2\big]\phi \\ &= \; -\mathrm{sgn}(\phi)\kappa\tau^3/2 + e\tau(\tau-c). \end{aligned} \tag{4.9}$$

We will be applying Lemma 4.3 to the system formed by (4.9) and (4.5) (modifying (4.9) appropriately). This system has a solution set identical to that of (4.5)–(4.6) (certainly on intervals not containing 0, c and $2c$, and by a continuity argument, on any interval). To compute (4.7) in this case, note that p, given as a ratio of the coefficient of ϕ to that of ϕ' in (4.9), has a partial fraction decomposition of the form $p = m/(\tau - c) - 1/(\tau - 2c) + b/\tau^2 - (2m-1)/\tau$. Similarly, $q = \mathrm{sgn}(\phi)\kappa/(2(\tau - c)) + (e - 2\mathrm{sgn}(\phi)\kappa c)/(2c(\tau - 2c)) - e/(2\tau c)$. Using $A = \tau^2(\tau - c)^2$, $B = (\tau - c)[m\tau^2 - (\tau - c)(2(m-1)\tau - b)]$, $C = -m\tau^2$ and $D = -\mathrm{sgn}(\phi)k\tau^2/2$, two long but direct computations gives

$$\begin{aligned} D - A(q' - pq) - Bq \;&=\; 0, \\ A(p^2 - p') - Bp + C \;&=\; -2bc(\tau-c)^2/(\tau(\tau-2c)). \end{aligned} \tag{4.10}$$

This immediately gives

Proposition 4.4. *Suppose $bc \neq 0$. Then the system (4.5)–(4.6) has no nonzero solutions on any nonempty open interval.*

Proof. Assume $bc \neq 0$. Then neither side of the second of equations (4.10) vanishes identically on the given interval. Hence Lemma 4.3 implies that any solution is the ratio of the left-hand sides of two equations in (4.10), away from the point c. This ratio is the zero function. By continuity, neither the system formed by (4.9) and (4.5), nor the system (4.5)–(4.6), admits any nonzero solutions on the given interval. □

4.4. **Solutions for the case $c = 0$.** If $c = 0$, equations (4.5)–(4.6) take the form,

$$\begin{aligned} \tau^4\phi'' \;+\; [(2-m)\tau^3 + b\tau^2]\,\phi' \;-\; m\tau^2\phi \;&=\; -\mathrm{sgn}\,(\phi)\kappa\,\tau^2/2, \\ -\tau^4\phi'' \;+\; [((m-1)\tau^3 - b\tau^2]\phi' \;+\; b\tau\phi \;&=\; e\tau, \end{aligned} \tag{4.11}$$

with m a positive integer, and $b \neq 0$. As special solutions, one can take $\mathrm{sgn}(\kappa)\kappa/(2m)$ for the first, and e/b for the second. A basis of solutions to each associated homogeneous equation is given by $\{\tau^m \exp(b/\tau), \sum b^{m-l}\tau^l/(m-l)!\}$, where the sum ranges over $l = 0 \dots m-1$ for the first, and $l = 1 \dots m-1$ for the second. Thus, the general solution to the system has the form

$$\phi = A + B\sum_{l=1}^{m-1} \frac{b^{m-l}}{(m-l)!}\tau^l + C\tau^m \exp(\,b/\tau) \tag{4.12}$$

for arbitrary constants A, B and C (where A represents the sum of an arbitrary multiple of $b^m/m!$ with $\mathrm{sgn}(\kappa)\kappa/(2m) + e/b$).

5. Geometry and duality for SKR metrics

5.1. **Local geometry of SKR metrics.** We recall here the main case in the geometric classification of SKR metrics. Let $\pi : (L, \langle\cdot,\cdot\rangle) \to (N, h)$ be a Hermitian holomorphic line bundle over a Kähler-Einstein manifold of complex dimension $m-1$. Assume that the curvature of $\langle\cdot,\cdot\rangle$ is a multiple of the Kähler form of h. Note that, if N is compact and h is not Ricci flat, this implies that L is smoothly isomorphic to a rational power of the anti-canonical bundle of N.

Consider, on $L\smallsetminus N$ (the total space of L excluding the zero section), the metric g given by

$$g|_{\mathcal{H}} = 2|\tau - c|\,\pi^* h, \quad g|_{\mathcal{V}} = \frac{Q(\tau)}{(ar)^2}\,\mathrm{Re}\,\langle\cdot,\cdot\rangle, \tag{5.1}$$

where
– $\mathcal{V}, \mathcal{H}$ are the vertical/horizontal distributions of L, respectively, the latter determined via the Chern connection of $\langle\cdot,\cdot\rangle$,
– $c, a \neq 0$ are constants,
– r is the norm induced by $\langle\cdot,\cdot\rangle$,
– τ is a function on $L\smallsetminus N$, obtained by composing with r another function, denoted via abuse of notation by $\tau(r)$, and obtained as follows: one fixes an open interval I and a positive C^∞ function $Q(\tau)$ on I, solves the differential equation $(a/Q)\,d\tau = d(\log r)$ to obtain a diffeomorphism $r(\tau) : I \to (0, \infty)$, and defines $\tau(r)$ as the inverse of this diffeomorphism.

The pair (g, τ), with $\tau = \tau(r)$, is an SKR pair (see [6, §8 and §16], as well as [7, §4]), and $|\nabla\tau|_g^2 = Q(\tau(r))$. If g is nontrivial, the connection on L will not be flat. The constant κ of Remark 3.9 is the Einstein constant of h, so that if g is nontrivial, it is in fact standard (for an arbitrary SKR metric, h need not be Einstein if $m = 2$). For g standard, or merely nontrivial, the SKR constant c (see again Remark 3.9) coincides with c of (5.1).

Conversely, for any standard nontrivial SKR metric (M, J, g, τ), any point in M_τ has a neighborhood biholomorphically isometric to an open set in some triple $(L\smallsetminus N, g, \tau(r))$ as above (this is a special case of [6, Theorem 18.1]). This biholomorphic isometry identifies span$\,(\nabla\tau, J\nabla\tau)$ and its orthogonal complement, with $\mathcal{V}$ and, respectively, $\mathcal{H}$. Moreover, one can extend (some) $(g, \tau(r))$ to all of L, and then such a biholomorphic isometry can also be defined on neighborhoods of points in $M\smallsetminus M_\tau$ [7, Remark 16.4].

5.2. **Duality for SKR metrics.** By Proposition 3.5, an SKR pair (g, τ) satisfies a Ricci-Hessian equation (2.4) at points of the noncritical set M_τ in which $\nabla d\tau$ is not a multiple of g. On this set (with $\tau^{-1}(0)$ excluded), the involution of §2.2 yields a new pair $(\widehat{g}, \widehat{\tau})$, which also satisfies a Ricci-Hessian equation. In general, not much can be said about $\widehat{g}$. However, a special case of the affine change mentioned in Remark 2.3 involves changing τ by an additive constant. This produces a new Killing potential t, with (g, t) an SKR pair very closely related to (g, τ). If the additive constant is chosen to be minus the SKR constant c, applying the involution to (g, t) yields metrics which are Kähler with respect to an oppositely oriented complex structure. In fact, they are even SKR metrics. We provide a proof of this in the following proposition, for the sake of completeness. Similar less detailed statements appear in [8, Remark 28.4] and [1, end of §5.5 and §5.6].

Proposition 5.1. *Let g be a standard SKR metric on (M, J), with Killing potential τ and corresponding SKR constant c. If $t = \tau - c$, then the associated pair $(\widehat{g}, \widehat{t}) = (g/t^2, 1/t)$ is a standard SKR pair on $M \smallsetminus \tau^{-1}(c)$.*

In fact, the proof will imply that the metric $\widehat{g}$ is Kähler with respect to the complex structure $\bar{J}$ given by $\bar{J}|_{\mathcal{H}} = J|_{\mathcal{H}}$, $\bar{J}|_{\mathcal{V}} = -J|_{\mathcal{V}}$, where $\mathcal{H}$ is the orthogonal complement of $\mathcal{V} = \operatorname{span}(\nabla\tau, J\nabla\tau)$. This structure, defined on M_τ, extends uniquely to M (see [7, Remark 16.4]), and the corresponding extension of $\widehat{g}$ (see end of §5.1) to $M \smallsetminus \tau^{-1}(0)$ is still Kähler with respect to it.

Proof. By the classification of SKR metrics, it is enough to consider M as a subset of the model line bundle L of §5.1. For simplicity, we take $M = L \smallsetminus N$. On L, the complex structure $\bar{J}$ defines the complex conjugate bundle structure, which we denote $\bar{L}$. We will show that the metric $\widehat{g}$ is an SKR metric, by constructing it explicitly as in §5.1, but on the line bundle $\bar{L}$. This line bundle is smoothly isomorphic to the dual bundle L^*, and hence the construction will transfer to a holomorphic line bundle, which is one of the requirements for the data used in §5.1. The proof that such structures are Kähler is indicated in [6, §16] (or, quite efficiently, via the methods in [1]).

The metric $\widehat{g}$ is obtained from the model metric g as follows: first replace $\langle\cdot,\cdot\rangle$, a, τ and I, respectively, with the complex conjugate fiber metric $\overline{\langle\cdot,\cdot\rangle}$, the constant $\widehat{a} = -a$, the function $\widehat{t} = 1/(\tau - c)$ and the open interval $\widehat{I}$ which is the image of the decreasing diffeomorphism $I \ni \tau \to \widehat{t} \in \widehat{I}$. We then replace c by $\widehat{c} = 0$, and have Q replaced with a function $\widehat{Q}$ which is a solution to the equation $a\,\widehat{Q}/Q = \widehat{a}\,d\widehat{t}/d\tau$. Finally, using these new data, along with h, r and $\mathcal{H}$, one defines a new standard SKR metric exactly as in (5.1). Note that the definition of $\widehat{Q}$ guarantees that the required relation $(\widehat{a}/\widehat{Q})\,d\widehat{t} = d(\log r)$ holds, and positivity of $\widehat{Q}$ follows from its defining equation together with the fact that $\widehat{t}(\tau)$ is decreasing. To conclude that this standard SKR metric is indeed $\widehat{g} = g/(\tau - c)^2$, one computes its two factors to be $2|\widehat{t} - \widehat{c}| = 2/|\tau - c|$ and $\widehat{Q}(\widehat{t})/(\widehat{a}\,r)^2 = -[Q(\tau)/(a\,r)^2]\,d\widehat{t}/d\tau = Q(\tau)/[(a\,r)^2(\tau - c)^2]$. □

Remark 5.2. In the case $c = \widehat{c}$, i.e. $c = 0$ we have $t = 1/\tau$, so that $(\widehat{g}, t) = (\widehat{g}, \widehat{\tau})$. Then, by Remark 3.9, one has $Q = 2\tau\phi$, and similarly for $\widehat{Q}$. Hence $\widehat{\phi}/\phi = \tau\widehat{Q}/(tQ) = -(\tau/t)\,dt/d\tau = -(\tau/(1/\tau)) \cdot (-1/\tau^2) = 1$, i.e. $\widehat{\phi}(t) = \phi(\tau)$. The same conclusion can be reached without the use of the geometric description of SKR metrics in §5.1, by restricting (2.2) to $\mathcal{H}$ and using (3.1) and Remark 3.9.

Remark 5.3. Still assuming $c = 0$, and using all the above conventions, suppose one fixes all the data defining g in (5.1), except for $Q = 2\tau\phi$, which varies only by changing $\phi(\tau)$ in the solution space of equation (4.1). If, in these circumstances, for each such solution, ϕ' satisfies the requirement in Remark 4.2, it follows that (2.4), and in particular, $\alpha = \alpha(\tau)$ does not vary for all these metrics. As they all share the same associated equation (4.1), the corresponding dual metrics $\widehat{g}$ also share their own associated equation (4.1), written with $t = \widehat{\tau}$ and $\widehat{\alpha}$, the latter determined as in (2.5). Since the solution space determines the coefficients of a linear differential equation, the result $\widehat{\phi}(t) = \phi(\tau)$ now implies that (4.1) for $(\widehat{g}, \widehat{\tau})$ is obtained from (4.1) of (g, τ) simply by the change of variable $\tau \to \widehat{\tau} = 1/\tau$.

6. Proof of Theorem A

Theorem 6.1. *Given a standard SKR pair (g, τ), if $\widehat{g} = g/\tau^2$ is a non-Einstein Ricci soliton with soliton function a multiple of τ^{-1}, then $\widehat{g}$ is Kähler, and locally of the type given by Koiso in* [10] *(or Cao in* [4]*).*

Proof. In fact, as the pair (g, τ) is standard, the associated function ϕ cannot be identically zero (see Remark 3.8). As the premises are those of §4.2, as a function on the image of τ, the function ϕ solves the system (4.5)–(4.6), but can do so only if $bc = 0$, by Proposition 4.4 (here $b\tau^{-1}$ denotes, as in §4.2, the soliton function). As $\widehat{g}$ is a non-Einstein Ricci soliton, $b \neq 0$ (Remark 2.5). Hence the SKR constant c is zero. This implies, by Proposition 5.1 and the paragraph past it, that the soliton $\widehat{g}$ is Kähler on $M \smallsetminus \tau^{-1}(0)$, with respect to a complex structure oppositely oriented to that with respect to which g is Kähler. By Remark 5.2, $\widehat{\phi}(\widehat{\tau}) = \phi(\tau)$, so that, by (4.12) and the definition of c in Remark 3.9,

$$\widehat{Q} = 2\widehat{\tau}\widehat{\phi} = \frac{2}{\tau}\left[A + B\sum_{l=1}^{m-1} \frac{b^{m-l}}{(m-l)!}\frac{1}{\tau^l} + C\frac{1}{\tau^m}\exp(b\tau)\right],$$

for some constants A, B, C. It is known (cf. [3, §2]) that a metric $\widehat{g}$ with the characteristics given in §5.1, and such an expression for $\widehat{Q}$, is (locally) of the form found by Koiso. □

We end with the

Proof of Theorem A. Let M be of dimension at least six, with (g, τ) a pair for which g is Kähler. If the associated pair $(\widehat{g}, \widehat{\tau})$ is a Ricci soliton, then, by Corollary 3.7, (g, τ) is an SKR pair. (This will also hold in dimension four if Q and $\Delta\tau$ are τ-dependent, see the paragraph before Proposition 3.3.) The non-reducibility assumption on g implies, in these dimensions, that it is a standard SKR metric. As the metric $\widehat{g}$ cannot be Einstein by Remark 2.5, Theorem 6.1 implies that $\widehat{g}$ is Kähler (and locally of the type given by Koiso).

As to the alternative version of this theorem, mentioned in the Introduction, if τ is a Killing potential and the soliton function is nonconstant, smooth and τ-dependent, then it is in fact nonconstant and *affine* in τ^{-1}, by Proposition 3.1. But then the soliton equation also holds with a soliton function which is just a nonconstant multiple of τ^{-1}. The theorem then follows just as in the proof above.

Acknowledgements

The author thanks A. Derdzinski, who initially participated in this work, and provided encouragement to carry on with it even after deciding not to coauthor it. He also read various versions, and made numerous suggestions. These greatly improved the style, and contributed significantly to the mathematical content of this paper, most notably with regard to the notion of duality. The author also thanks the reviewer for various remarks, especially the suggestion to make explicit the alternative version of the main theorem.

References

[1] Apostolov, V., Calderbank, D.M.J., Gauduchon P.: Hamiltonian 2-forms in Kähler geometry, I. General theory. J. Differential Geom. 73 (2006), 359–412

[2] Apostolov, V., Calderbank, D.M.J., Gauduchon P., Tønnesen-Friedman C. W.: Hamiltonian 2-forms in Kähler geometry, II. Global classification. J. Differential Geom. 68 (2004), 277–345

[3] Apostolov, V., Calderbank, D.M.J., Gauduchon P., Tønnesen-Friedman C. W.: Hamiltonian 2-forms in Kähler geometry, IV. Weakly Bochner-flat Kähler manifolds. In: arXiv:math.DG/0511119, to appear in Comm. Anal. Geom.

[4] Cao H.-D.: Existence of gradient Kähler-Ricci solitons. In: Elliptic and parabolic methods in geometry, Minneapolis MN 1994, 1–16 A. K. Peters, Wellesley MA (1996)

[5] Chen X., LeBrun C., Brian Weber B.: On Conformally Kähler, Einstein Manifolds. In: J. Amer. Math. Soc., posted on Jan. 28, 2008, PII: S 0894-0347(08)00594-8 (to appear in print)

[6] Derdzinski, A., Maschler, G.: Local classification of conformally-Einstein Kähler metrics in higher dimensions. Proc. London Math. Soc. 87 (2003), 779–819

[7] Derdzinski, A., Maschler, G.: Special Kähler-Ricci potentials on compact Kähler manifolds. J. reine angew. Math. 593 (2006), 73–116

[8] Derdzinski, A., Maschler, G.: A moduli curve for compact conformally-Einstein Kähler manifolds. Compositio Math. 141 (2005), 1029–1080

[9] Hamilton, R.S.: The Ricci flow on surfaces. In: Mathematics and general relativity, Santa Cruz CA, 1986. Contemp. Math., vol. 71, pp. 237–262. AMS, Providence RI (1988)

[10] Koiso N.: On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics. In: Recent topics in differential and analytic geometry, Adv. Stud. Pure Math., vol. 18-I, pp. 327–337. Academic Press, Boston MA (1990)

[11] Perelman G.: The entropy formula for the Ricci flow and its geometric applications. In: arXiv:math.DG/0211159

[12] Wang, X.-J.; Zhu, X.: Kähler-Ricci solitons on toric manifolds with positive first Chern class. Adv. Math. 188 (2004), 87–103

Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322, U.S.A.

E-mail address: gm@mathcs.emory.edu