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SPECIAL KÄHLER-RICCI POTENTIALS ON COMPACT KÄHLER MANIFOLDS

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Abstract. By a *special Kähler-Ricci potential* on a Kähler manifold we mean a nonconstant real-valued C^∞ function τ such that $J(\nabla\tau)$ is a Killing vector field and, at every point with $d\tau \neq 0$, all nonzero tangent vectors orthogonal to $\nabla\tau$ and $J(\nabla\tau)$ are eigenvectors of both $\nabla d\tau$ and the Ricci tensor. For instance, this is always the case if τ is a nonconstant C^∞ function on a Kähler manifold (M,g) of complex dimension $m > 2$ and the metric $\tilde{g} = g/\tau^2$, defined wherever $\tau \neq 0$, is Einstein. (When such τ exists, (M,g) may be called *almost-everywhere conformally Einstein.*) We provide a complete classification of compact Kähler manifolds (M,g) with special Kähler-Ricci potentials, showing, in particular, that in any complex dimension $m \geq 2$ they form two separate classes: in one, M is the total space of a holomorphic $\mathbf{C}\mathrm{P}^1$ bundle; in the other, M is biholomorphic to $\mathbf{C}\mathrm{P}^m$. We then use this classification to prove a structure theorem for compact Kähler manifolds of any complex dimension $m > 2$ which are almost-everywhere conformally Einstein.

§1. Introduction

The present paper, although of independent interest, is the second in a series of three papers that also includes [**7**] and [**8**].

We call τ a *special Kähler-Ricci potential* [**7**] on a Kähler manifold (M,g) if

(1.1) τ is a nonconstant Killing potential on (M,g) and, at every point with $d\tau \neq 0$, all nonzero tangent vectors orthogonal to $v = \nabla\tau$ and to $u = Jv$ are eigenvectors of both $\nabla d\tau$ and the Ricci tensor r.

Of our two main results, one (Theorem 16.3) provides a complete classification of compact Kähler manifolds (M,g) with special Kähler-Ricci potentials.

The other main result is a structure theorem for, and a partial classification of, those compact Kähler manifolds (M,g) in complex dimensions $m \geq 3$ which are *almost-everywhere conformally Einstein* in the sense of (1.2) below; for $m = 2$ the same argument is valid under the stronger assumption (1.3). As outlined later in this section, the second main result is used in [**8**] to obtain a complete classification of compact Kähler manifolds satisfying (1.2) in complex dimensions $m \geq 3$, or (1.3) for $m = 2$.

Our interest in (1.1) was in fact sparked by its being related to the almost-everywhere conformally Einstein case. Specifically, we consider two conditions.

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(1.2) (M,g) is a Kähler manifold of complex dimension m and τ is a nonconstant C^∞ function on M such that the conformally related metric $\tilde{g} = g/\tau^2$, defined wherever $\tau \neq 0$, is Einstein.

(1.3) M, g, m, τ satisfy (1.2) and $d\tau \wedge d\Delta\tau = 0$ everywhere in M.

The additional clause in (1.3) states that locally, at points with $d\tau \neq 0$, the Laplacian of τ is a function of τ. In [**7**], Corollary 9.3, we found that

(1.4) condition (1.2) with $m \geq 3$, or (1.3) with $m = 2$, implies (1.1).

As shown in [**7**], Proposition 6.4, (1.3) follows from (1.2) when $m > 2$. In other words, (1.3) is really stronger than (1.2) only for Kähler surfaces ($m = 2$).

Furthermore, (1.1) is closely related, although not equivalent, to the requirement (see [**7**], §7) that $\nabla d\tau + \chi\,\mathrm{r} = \sigma g$ for some C^∞ functions χ and σ, reminiscent of Kähler-Ricci solitons (cf. Remark 5.3 below). Functions τ appearing in (1.2) also arise from *Hamiltonian* 2*-forms* [**2**], which, on compact Kähler manifolds, were recently classified by Apostolov et al. [**3**].

We now proceed to describe in more detail our two main results. The first of them deals with triples (M, g, τ) formed by a compact Kähler manifold (M, g) of complex dimension $m \geq 1$ and a special Kähler-Ricci potential $\tau : M \to \mathbf{R}$. In §5 and §6 we construct two classes of such triples, labelled 1 and 2:

- In Class 1, M is the total space of a suitable holomorphic $\mathbf{C}\mathrm{P}^1$ bundle over a compact Kähler manifold (N, h) which is also Einstein unless $m = 2$.
- In Class 2, M is biholomorphic to $\mathbf{C}\mathrm{P}^m$.

In both classes, M is obtained from the total space of a holomorphic line bundle by a compactification, projective (in Class 1), or one-point (in Class 2, for the dual tautological bundle over $\mathbf{C}\mathrm{P}^{m-1}$). The metric g is chosen so that, in particular, the line-bundle projection is a horizontally homothetic submersion [**9**] with totally geodesic fibres. A direct characterization of Classes 1 and 2 is described below.

A prominent ingredient of the constructions just mentioned is a C^∞ function $\tau \mapsto Q$ on an interval $[\tau_{\min}, \tau_{\max}]$, subject to specific positivity and boundary conditions (listed in (5.1)), but otherwise arbitrary. Substituting for the independent variable τ the special Kähler-Ricci potential τ on the resulting Kähler manifold (M, g), one turns Q into a function $M \to \mathbf{R}$, which then equals $|\nabla\tau|^2$. This major role of $|\nabla\tau|^2$ as a function of τ is common in constructions of Killing potentials; see, for instance, [**11**] and [**18**].

Theorem 16.3, in turn, classifies all compact Kähler manifolds (M, g) with special Kähler-Ricci potentials τ. It states that, up to biholomorphic identifications, every such triple (M, g, τ) must belong to one of the two classes constructed in §5 and §6. Note that, due to arbitrariness of the function $\tau \mapsto Q$, if pairs (g, τ) satisfying (1.1) on a given compact complex manifold M exist at all, they must form an infinite-dimensional moduli space. In this regard, (1.1) differs from (1.2) (in complex dimensions $m \geq 3$) or (1.3); see [**7**].

Classes 1 and 2 also have an intrinsic characterization. Namely, any special Kähler-Ricci potential τ on a compact Kähler manifold (M, g) has exactly two critical manifolds; one of them is of complex codimension one, the other may have complex codimension one (in Class 1) or consist of a single point (in Class 2). The fact just stated, established in Proposition 11.5, constitutes a major step in the classification argument that we use to prove Theorem 16.3.

To provide at least a partial explanation of "why" Theorem 16.3 is true, we now briefly summarize the steps leading to its proof. (A more detailed description is given eight paragraphs below.) We start with a special Kähler-Ricci potential τ on a compact Kähler manifold (M,g) of complex dimension $m \geq 2$. First, we show that, as mentioned above, τ has two critical manifolds, N and N^*, with $\dim_{\mathbf{C}} N^* = m-1$ and either $\dim_{\mathbf{C}} N = m-1$, or $\dim_{\mathbf{C}} N = 0$. We then select a specific punctured-disk subbundle $\mathcal{L}'$ of the normal bundle $\mathcal{L}$ of N and prove that the normal exponential mapping Exp sends $\mathcal{L}'$ diffeomorphically onto $M' = M \smallsetminus (N \cup N^*)$. Next, we exhibit a fibre-preserving diffeomorphism $\varPhi : \mathcal{L} \smallsetminus N \to \mathcal{L}'$ that multiplies each nonzero normal vector z by a factor depending only on $|z|$, and show that $\mathrm{Exp} \circ \varPhi$ is a biholomorphism $\mathcal{L} \smallsetminus N \to M'$, admitting an extension to a biholomorphism between a suitable compactification of $\mathcal{L}$ and M. Finally, the pullbacks of g and τ under that extension are verified to coincide with the objects constructed as in §5 or §6 from data which our M, g and τ naturally distinguish on $\mathcal{L}$.

Theorem 16.3 leads, via (1.4), to our second main result, consisting of Theorems 17.4, 18.1, 19.3 and Corollary 19.4. They form a structure theorem for, and a partial classification of, quadruples (M, g, m, τ) with compact M that satisfy (1.2) for $m \geq 3$, or (1.3) for $m = 2$. Specifically, in §17 all such quadruples, with compact M, are divided into six disjoint "types" (a1), (a2), (b1), (b2), (c1), (c2), the digit 1 or 2 indicating in which of our Classes 1 and 2 the type is contained. We then prove that types (a2), (b1), (b2) are empty (Theorem 17.4), and type (c2) leads to a conclusion (Corollary 19.4) which, as shown in [**8**], cannot be satisfied; therefore, type (c2) eventually turns out to be empty as well.

Thus, all compact Kähler manifolds of complex dimensions $m \geq 2$ which are almost-everywhere conformally Einstein (and, if $m = 2$, also satisfy the additional clause in (1.3)) belong to type (a1) or (c1), and hence to Class 1. We emphasize that the proof of this fact uses not only results of the present paper, but also those of [**8**].

The remaining two parts of our structure and partial-classification result for quadruples (M, g, m, τ) as above are Theorems 18.1 and 19.3. The former classifies type (a1); every M occurring there is a flat holomorphic $\mathbf{C}\mathrm{P}^1$ bundle. The latter reduces the classification of type (c1) (in which M always is a *nonflat* holomorphic $\mathbf{C}\mathrm{P}^1$ bundle) to the question of finding all rational functions of one real variable that lie in a specific three-dimensional vector space depending on m and satisfy positivity and boundary conditions closely related to those in (5.1).

An answer to this last question is given in [**8**], where we classify type (c1) by dividing it into three disjoint families: one, discovered by Page [**17**] for $m = 2$ and, for $m \geq 3$, by Bérard Bergery [**4**]; another, that includes some known Kähler surface metrics [**10**], [**20**] along with some new metrics in all higher dimensions; and a new, third family, present only in odd complex dimensions $m \geq 9$. Type (a1) appears in [**8**] as a fourth family, characterized by local reducibility of the Kähler metrics g.

In the last three families, τ with (1.2) vanishes somewhere in M, giving rise to examples of *conformally compact* Einstein manifolds, cf. [**1**].

To describe how the paper is organized, it is convenient to divide the text into three parts. The first (Sections 3 – 6) contains descriptions of examples, leading up to the constructions of our Classes 1 and 2. In the second part (Sections 7 – 16) we prove Theorem 16.3. The third part is devoted to results about compact Kähler manifolds which are almost-everywhere conformally Einstein. Further details concerning the three parts are provided in the following three paragraphs.

In the first part we use a local construction developed in [**7**], along with a standard compactification argument.

The second part, that is, our proof of Theorem 16.3, involves four major steps, appearing in Sections 7, 11, 13 and 15. The first step is Proposition 7.3, stating that, for a special Kähler-Ricci potential τ on a Kähler manifold of complex dimension m, any critical manifold of τ must be of complex dimension $m-1$ or 0, while the Hessian of τ at any critical point has only one nonzero eigenvalue (and hence is semidefinite). Note that this is a local result; we prove it by analyzing the structure of $\nabla d\tau$ near a critical point. In the next step, Proposition 11.5, we show that, if M is also assumed compact, τ must have exactly two critical manifolds, and $|\nabla\tau|^2$ is a C^∞ function of τ satisfying the positivity and boundary conditions (5.1). That there are just two critical manifolds follows since, due to the semidefiniteness of its Hessian, τ is a Morse-Bott function having a local extremum at every critical point. Proposition 11.5 allows us to introduce the intrinsic definition of Classes 1 and 2, mentioned earlier in this section, as the case of two isolated critical points is easily excluded when $m>1$. Step three, Lemma 13.2, describes a "large" tubular neighborhood of a critical manifold N of τ, still assuming that M is compact; namely, the normal exponential mapping Exp of N is shown to be a diffeomorphism between a specific open-disk subbundle of the normal bundle of N and the complement, in M, of the other critical manifold. In the last step, Lemma 15.1, we prove several properties of the differential of Exp needed for the final conclusion in the proof of Theorem 16.3. The conclusion in question states that a specific mapping between our (M,g) and the underlying Kähler manifold of a Class 1 or Class 2 triple, constructed (as in §5 or §6) from ingredients naturally provided by (M,g) and τ is, in fact, a biholomorphic isometry. More precisely, we select N so that *the other* critical manifold is of complex dimension $m-1$, where $m=\dim_{\mathbf{C}} M$, and the choice between Class 1 and Class 2 depends on whether $\dim_{\mathbf{C}} N$ is $m-1$ or 0.

The third part begins with §17. Its starting point, Proposition 17.1, is a local result proved in [**7**], and states that the assertion about $Q=|\nabla\tau|^2$ being a C^∞ function of τ (mentioned above as a consequence of (1.1) when M is compact), remains true even without compactness of M, as long as, instead of (1.1), one uses the stronger assumption (1.2) with $m\geq 3$, or (1.3) with $m=2$. In addition, the function $\tau\mapsto Q=|\nabla\tau|^2$ then is *rational* and must lie in one of three specific sets of rational functions. It is by pairing up each of the three sets with Classes 1 and 2, for compact M, that we arrive at the six types (a1) – (c2) discussed earlier.

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*For instance, what [**7**] refers to as §36 of this paper has now become Remark 16.4.

§2. Preliminaries

Our notational conventions include the following:

$$
\begin{aligned}
&\text{i)} && R(u,v)w \,=\, \nabla_v\nabla_u w \,-\, \nabla_u\nabla_v w \,+\, \nabla_{[u,v]}w\,,\\
&\text{ii)} && \nabla v : TM \to TM \quad \text{with} \quad (\nabla v)w \,=\, \nabla_w v \quad \text{for} \;\; w\in T_xM \;\; \text{and} \;\; x\in M,\\
&\text{iii)} && (\nabla d\tau)(u,w) \,=\, g(\nabla_u v, w) \quad \text{for} \;\; v=\nabla\tau, \;\; \text{and} \;\; \Delta\tau \,=\, \mathrm{Trace}_g(\nabla d\tau),\\
(2.1)\quad &\text{iv)} && \omega(u,v) \,=\, g(Ju,v)\,, \qquad \text{for} \;\; u,v\in T_xM \;\; \text{and} \;\; x\in M,\\
&\text{v)} && \mathcal{L} \,=\, \{(y,z) : y\in N, \;\; z\in\mathcal{L}_y\} \qquad \text{and} \qquad N\subset\mathcal{L}\,,\\
&\text{vi)} && R(u,v)w \,=\, i\,\Omega(u,v)w\,,\\
&\text{vii)} && \Omega = i\,d\Gamma \;\text{for}\; \Gamma \;\text{with}\; \nabla_v w = \Gamma(v)\,w.
\end{aligned}
$$

Specifically, in (i), R is the curvature tensor of a (linear) connection ∇ in any real/complex vector bundle over a manifold, u, v are C^2 vector fields tangent to the base and w is a C^2 section of the bundle; in (ii), ∇ is a connection in the tangent bundle TM of a manifold M, while v is a C^1 vector field on M and ∇v is its covariant derivative, treated as a vector-bundle morphism; in (iii), $\nabla d\tau$ is the second covariant derivative of a C^2 function τ on a Riemannian manifold, g is the metric, u, w are any tangent vector fields, and the symbol ∇ stands both for the Levi-Civita connection and the gradient, while Δ is the Laplacian; in (iv), ω is the Kähler form of a Kähler manifold (M,g) and J denotes the complex-structure tensor of its underlying complex manifold; in (v), $\mathcal{L}$ denotes both a vector bundle over a manifold N and its total space, while N is identified with the zero section; finally, in (vi) and (vii), the complex-valued 2-form Ω and 1-form Γ are the *curvature form* and *connection form* of any C^∞ connection ∇ in a complex line bundle $\mathcal{L}$ over a manifold N, while in (vi) u, v, w, R are as in (i), and in (vii) v, w are local C^∞ sections of TN and $\mathcal{L}$, the latter without zeros.

Remark 2.1. As usual, a real-valued C^∞ function τ on a Kähler manifold (M,g) is said to be a *Killing potential* if $u = J(\nabla\tau)$ is a Killing field on (M,g). One has the well-known equality $dY = -2\,\mathrm{r}(\nabla\tau,\,\cdot\,)$, where $Y = \Delta\tau$, cf. (2.1.iii), and r is the Ricci tensor. (See, for instance, [**7**], formula (5.4).)

We call a (real) C^∞ vector field v on a complex manifold *holomorphic* if $\boldsymbol{L}_v J = 0$, where $\boldsymbol{L}$ is the Lie derivative. For a C^∞ vector field v on a *Kähler* manifold (M,g), this is the case if and only if J and ∇v commute (cf. [**7**], formula (5.1)).

The following lemma is also well known; see, for instance, [**7**], Lemma 5.3.

Lemma 2.2. *Let (M,g) be a Kähler manifold. For every Killing potential τ on (M,g), the Killing field $J(\nabla\tau)$ is holomorphic. Conversely, if $H^1(M,\mathbf{R}) = \{0\}$, then every holomorphic Killing vector field on (M,g) has the form $J(\nabla\tau)$ for a Killing potential τ, which is unique up to an additive constant.* ■

Let $\tau : M \to \mathbf{R}$ be a C^∞ function on a manifold M. If all connected components N of the set $\mathrm{Crit}(\tau)$ of its critical points happen to satisfy conditions (a), (b) in Remark 2.3(iii) below, we will refer to them as the *critical manifolds* of τ.

Remark 2.3. Given a nonconstant Killing potential τ on a Kähler manifold (M,g), let $M' \subset M$ be the open set on which $d\tau \neq 0$. Then

(i) $\nabla d\tau \neq 0$ wherever $d\tau = 0$,

(ii) M' is connected and dense in M,

(iii) for every connected component N of the set of critical points of τ,

(a) N is contained in an open set that intersects no other component,

(b) $N \subset M$ is a closed set and a submanifold with the subset topology,

(c) N is totally geodesic in (M,g) and $\dim M - \dim N \geq 2$,

(d) $T_yN = \operatorname{Ker}[(\nabla v)(y)] = \{w \in T_yM : \nabla_w v = 0\}$ whenever $y \in N$, with $v = \nabla\tau$. Thus, N is a complex submanifold of M.

Namely, (d) in (iii) follows since $T_yN = \operatorname{Ker}[(\nabla u)(y)]$ for the Killing field $u = Jv$ (see [**7**], Lemma 12.2(d)), while $\nabla u = J \circ (\nabla v) = (\nabla v) \circ J$ as $\nabla J = 0$ and v is holomorphic, cf. Lemma 2.2. Finally, (ii) is obvious from (a) – (c) in (iii), which, along with (i), are in turn justified in [**7**], Remark 5.4 and Lemma 12.2.

Any Riemannian/Hermitian fibre metric $\langle\,,\rangle$ in a real/complex vector bundle $\mathcal{L}$ over a manifold N is determined by its *norm function* $\mathcal{L} \to [0,\infty)$, later denoted by r (or, sometimes, s), which assigns $|z| = \langle z,z\rangle^{1/2}$ to each $(y,z) \in \mathcal{L}$. We also treat $\langle\,,\rangle$ as a fibre metric in the vertical subbundle $\mathcal{V}$ of $T\mathcal{L}$ (by identifying $\mathcal{V}_{(y,z)}$ with $\mathcal{L}_y$). If $\mathcal{L}$ is a complex vector bundle, any fixed real number $a \neq 0$ gives rise to vertical vector fields v,u on $\mathcal{L}$ given by $v(y,z) = az$ and $u(y,z) = iaz$, and, clearly, $\langle v,v\rangle = \langle u,u\rangle = a^2r^2$ and $\operatorname{Re}\langle v,u\rangle = 0$.

Remark 2.4. Let $\mathcal{L}$ be a C^∞ complex line bundle over a complex manifold N, and let $\mathcal{H}$ be the horizontal distribution of a fixed C^∞ linear connection in $\mathcal{L}$ whose curvature form $\varOmega$ is real-valued and skew-Hermitian, that is, $\varOmega(Jv,v') = -\varOmega(v,Jv')$ for all $y \in N$ and $v,v' \in T_yN$. Then $\mathcal{L}$ admits a unique structure of a holomorphic line bundle over N such that $\mathcal{H}$ is invariant under the complex structure tensor $J : T\mathcal{L} \to T\mathcal{L}$.

In fact, let $\varGamma$ be as in (2.1.vii) for a fixed C^∞ local trivializing section w of $\mathcal{L}$, defined on a contractible open set $N' \subset N$. Using w to identify the portion $\mathcal{L}'$ of $\mathcal{L}$ lying over N' with $N' \times \mathbf{C}$, and writing down the parallel-transport equation in terms of $\varGamma$, we see that, for any $(y,z) \in \mathcal{L}'$ and $(w,\zeta) \in T_{(y,z)}\mathcal{L}'$, the $\mathcal{H}$ component of (w,ζ) is $(w, -\varGamma(w)z)$. Thus, w is holomorphic for a holomorphic-bundle structure in $\mathcal{L}'$ for which $\mathcal{H}$ is J-invariant if and only if $\varGamma$ is of type $(1,0)$, that is, the bundle morphism $\varGamma : TN' \to N' \times \mathbf{C}$ is complex-linear. While $\varGamma$ need not be of type $(1,0)$, a $(1,0)$ form $\tilde\varGamma$ on N' with $d\tilde\varGamma = -i\varOmega$ exists: namely, $\tilde\varGamma = \partial\varphi$ with $\varphi : N' \to \mathbf{R}$ such that $i\varOmega = \partial\bar\partial\varphi$. Now $\tilde\varGamma = \varGamma + d\varPhi$ for some C^∞ function $\varPhi : N' \to \mathbf{C}$, and so $\mathcal{H}$ is J-invariant for the holomorphic-bundle structure in $\mathcal{L}'$ that makes the section $\tilde w = e^\varPhi w$ holomorphic (since the connection form corresponding to $\tilde w$ is $\tilde\varGamma$). Any other C^∞ section of $\mathcal{L}'$ without zeros having a $(1,0)$ connection form must equal $e^\varPsi\tilde w$, with $\varPsi : N' \to \mathbf{C}$ holomorphic as $d\varPsi$ is of type $(1,0)$, so that the structure in question is unique.

Remark 2.5. Given a holomorphic line bundle $\mathcal{L}$ over a complex manifold N, let N^* stand for N treated as the zero section $N^* \subset \mathcal{L}^*$ in the dual bundle $\mathcal{L}^*$, cf. (2.1.v). We define the *inversion biholomorphism* $\mathcal{L} \smallsetminus N \to \mathcal{L}^* \smallsetminus N^*$ to be the assignment $(y,z) \mapsto (y,z^{-1})$, where $z^{-1} \in \mathcal{L}^*_y$ is the unique $\mathbf{C}$-linear functional $\mathcal{L}_y \to \mathbf{C}$ sending z to 1. The inversion biholomorphism clearly sends the horizontal distribution $\mathcal{H}$ of any C^∞ linear connection in $\mathcal{L}$ onto the horizontal distribution $\mathcal{H}^*$ of the corresponding dual connection in $\mathcal{L}^*$. It also sends any Hermitian fibre metric $\langle\,,\rangle$ in $\mathcal{L}$ onto the multiplicative inverse of its dual metric $\langle\,,\rangle^*$ in $\mathcal{L}^*$, that is, $\langle z^{-1}, z^{-1}\rangle^* = \langle z, z\rangle^{-1}$ whenever $y \in N$ and $z \in \mathcal{L}_y \smallsetminus \{0\}$.

§3. Basic properties and simplest examples

This section begin with some basic results on special Kähler-Ricci potentials, established in [**7**], and gathered here for easy reference. They are followed by four numbered examples of cases where a function on a Kähler manifold satisfying conditions seemingly weaker than (1.1) must in fact be a special Kähler-Ricci potential due to additional circumstances such as one-dimensionality, reducibility with factors of a special type, or a large symmetry group. Example 3.6 is of particular importance, since what it describes is precisely our Class 2, introduced, more explicitly, later in §6.

First, we have some basic facts. Given a special Kähler-Ricci potential τ on a Kähler manifold (M, g), let $M' \subset M$ be the open set on which $d\tau \neq 0$, and let the vector fields v, u on M, distributions $\mathcal{H}, \mathcal{V}$ on M', and a function $Q : M \to \mathbf{R}$ be defined by $v = \nabla\tau$, $u = Jv$, $\mathcal{V} = \mathrm{Span}\,\{v, u\}$, $\mathcal{H} = \mathcal{V}^\perp$, and $Q = |\nabla\tau|^2$. Due to the eigenvector clause of (1.1) and Hermitian symmetry of both r and $\nabla d\tau$ (cf. [**7**], Lemma 5.2), there exist C^∞ functions $\phi, \psi, \lambda, \mu : M' \to \mathbf{R}$ with

$$\begin{aligned} &\mathrm{r} = \lambda g \ \text{ and } \ \nabla d\tau = \phi g \ \text{ on } \ \mathcal{H}, \\ &\mathrm{r} = \mu g \ \text{ and } \ \nabla d\tau = \psi g \ \text{ on } \ \mathcal{V}, \\ &\mathrm{r}(\mathcal{H},\mathcal{V}) = (\nabla d\tau)(\mathcal{H},\mathcal{V}) = \{0\} \qquad \text{for } \ \mathcal{H},\mathcal{V} \ \text{ as above.} \end{aligned} \tag{3.1}$$

The last line states that $\mathcal{H}$ is both r-orthogonal and $\nabla d\tau$-orthogonal to $\mathcal{V}$. If $\dim_{\mathbf{C}} M = 1$, we set $\phi = \lambda = 0$. By (3.1) and [**7**], Lemmas 7.5, 11.1(b), on M',

$$\begin{aligned} &\text{a) } \nabla_w v \text{ equals } \phi w \text{ (or, } \psi w\text{) whenever } w \text{ is a section of } \mathcal{H} \text{ (or, of } \mathcal{V}\text{)}, \\ &\text{b) } dQ = 2\psi\, d\tau, \ \text{ that is, } \ \nabla Q = 2\psi v, \ \text{ and } \ \nabla\phi = 2(\psi - \phi)\phi v/Q, \\ &\text{c) } Y = 2\psi + 2(m-1)\phi \text{ for } Y = \Delta\tau, \\ &\text{d) } dY = -2\mu\, d\tau, \text{ where } Y = \Delta\tau. \end{aligned} \tag{3.2}$$

Lemma 3.1. *For a special Kähler-Ricci potential τ on a Kähler manifold (M, g), let Q, ϕ and M' be as above. Then*

(i) *either $\phi = 0$ identically on M', or $\phi \neq 0$ everywhere in M',*
(ii) *if $\phi \neq 0$ on M', there exists a constant c with $Q/\phi = 2(\tau - c)$ and $\tau \neq c$ everywhere in M',*
(iii) *a number $\varepsilon \in \{-1, 0, 1\}$ is uniquely defined by requiring that $\varepsilon = 0$ when $\phi = 0$ on M' and $\varepsilon = \mathrm{sgn}\,(\tau - c)$ on M' if $\phi \neq 0$ everywhere in M'.*

(Thus, $\varepsilon = 0$ when $\dim_{\mathbf{C}} M = 1$.) In fact, (i) and (ii) are proved in [**7**], Lemma 12.5; relation $\tau \neq c$ on M' (obvious since $Q/\phi = 2(\tau - c)$ and $Q \neq 0$ on M' by the definition of M') yields (iii) as M' is connected, cf. Remark 2.3(ii). ■

Example 3.2. In complex dimension $m = 1$ special Kähler-Ricci potentials are nothing else than nonconstant Killing potentials τ (the rest of (1.1) being vacuously true). When defined only up to an additive constant, they thus are, locally, in a one-to-one correspondence $\tau \mapsto u = J(\nabla\tau)$ with nontrivial Killing fields u. This is clear from Lemma 2.2: if $m = 1$, every Killing field u is holomorphic, as skew-adjointness of ∇u gives $\nabla u = \psi J$ for some function ψ, so that J and ∇u commute.

Example 3.3. Special Kähler-Ricci potentials τ on suitable product Kähler manifolds $(M, g) = (N, h) \times (S, \gamma)$ of complex dimensions $m \geq 2$ are obtained when (N, h) is a Kähler manifold with $\dim_{\mathbf{C}} = m - 1$, Einstein unless $m = 2$, and (S, γ) is an oriented Riemannian surface with a nonconstant Killing potential $\tau : S \to \mathbf{R}$, which we then treat as a function on $M = N \times S$, constant along the N factor. (The existence of such τ amounts, locally, to the existence of a nontrivial Killing field on (S, γ), cf. Lemma 2.2 and Example 3.2.) In fact, conditions (3.1) are satisfied by the N and S factor distributions $\mathcal{H}$ and $\mathcal{V}$ on M, along with $\phi = 0$, the Laplacian $\psi = \Delta\tau/2$ of $\tau/2$ in (S, γ), the function λ such that λh is the Ricci tensor of h, and the Gaussian curvature μ of γ. Namely, as $\nabla u = \psi J$ (Example 3.2), we get $\nabla d\tau = \psi\gamma$ from (2.1.iii) with g replaced by γ.

For the next example, we need a lemma.

Lemma 3.4. *Let two vector fields v, u on a Riemannian manifold (M, g) be linearly independent at every point, and let $\mathcal{H}$ be a distribution on M with $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \mathrm{Span}\,\{v, u\}$. If G is a group of isometries of (M, g) such that at every $x \in M$ the action on T_xM, via differentials, of the isotropy subgroup of G at x leaves $v(x)$, $u(x)$ and $\mathcal{H}_x$ invariant and acts transitively on the unit sphere in $\mathcal{H}_x$, then $\mathcal{H} = \mathcal{V}^\perp$ and all nonzero vectors in $\mathcal{H}$ are eigenvectors of every G-invariant symmetric twice-covariant tensor field b on M.*

In fact, for such b and x the functions $b(v(x), w)$, $b(u(x), w)$ and $b(w, w)$ of $w \in \mathcal{H}_x$ are constant on the unit sphere; the first two are also linear, so they must be zero. This yields the last claim and, applied to $b = g$, gives $\mathcal{H} = \mathcal{V}^\perp$. ■

Example 3.5. Special Kähler-Ricci potentials τ can also be constructed on the Kähler manifold (U, g), where U is a G-invariant nonempty connected open subset of a Hermitian vector space V with $\dim_{\mathbf{C}} V = m \geq 1$ and g is a G-invariant Kähler metric on U, while $G \approx \mathrm{U}(m)$ is the group of automorphisms of V preserving the Hermitian inner product $\langle\,,\rangle$. Namely, we may choose $\tau : M \to \mathbf{R}$ to be a Killing potential with $u = Jv$ for $v = \nabla\tau$, where ∇ is the g-gradient and u is the vector field on U with $u(x) = aix$ for any fixed $a \in \mathbf{R} \smallsetminus \{0\}$.

Namely, u is an infinitesimal generator of the center subgroup of G, and hence a G-invariant holomorphic Killing field on (U, g). Thus, τ exists and is unique up to an additive constant (cf. Lemma 2.2). Applying Lemma 3.4 to the distribution $\mathcal{V}$ on $V \smallsetminus \{0\}$, with $\mathcal{V}_x = \mathbf{C}x$ and its $\langle\,,\rangle$-orthogonal complement $\mathcal{H}$, we now see that $\mathcal{V}$ and $\mathcal{H}$ are g-orthogonal to each other and (1.1) holds.

Example 3.6. For an integer $m \geq 1$ and a fixed point $y \in \mathbf{C}\mathrm{P}^m$, let g be any G-invariant Kähler metric on $\mathbf{C}\mathrm{P}^m$, where G is the group of all biholomorphisms $\mathbf{C}\mathrm{P}^m \to \mathbf{C}\mathrm{P}^m$ that keep y fixed and preserve the Fubini-Study metric g_{FS}. A special Kähler-Ricci potential τ on $(\mathbf{C}\mathrm{P}^m, g)$ is obtained as follows: $G \approx \mathrm{U}(m)$ in view of the usual identification of $\mathbf{C}^m$ with an open dense set in $\mathbf{C}\mathrm{P}^m$, under which $0 = y$. The center of G, isomorphic to $\mathrm{U}(1)$, is thus generated by a nontrivial holomorphic Killing field u on $(\mathbf{C}\mathrm{P}^m, g)$, unique up to a factor; we choose a Killing potential $\tau : \mathbf{C}\mathrm{P}^m \to \mathbf{R}$ with $u = J(\nabla\tau)$, where ∇ is the g-gradient. Such τ exists and is unique up to an additive constant (Lemma 2.2). That τ is a special Kähler-Ricci potential on $(\mathbf{C}\mathrm{P}^m, g)$ is immediate in view of Example 3.5.

§4. A local model

In this section we describe a local model for special Kähler-Ricci potentials: a construction that *yields, locally, up to local biholomorphic isometries, all special Kähler-Ricci potentials* τ *on Kähler manifolds at points with* $d\tau \neq 0$. The italicized statement is a classification result, proved in [**7**], Theorem 18.1.

Let there be given data $\mathcal{I}, \tau, Q, r, a, \varepsilon, c, m, N, h, \mathcal{L}, \mathcal{H}, \langle\,,\rangle$ consisting of an open interval $\mathcal{I} \subset \mathbf{R}$, a real variable $\tau \in \mathcal{I}$, a real constant $a \neq 0$, positive C^∞ functions Q, r of the variable $\tau \in \mathcal{I}$ with $dr/d\tau = ar/Q$, constants ε, c such that either $\varepsilon = 0$ (and c is left undefined), or $c \notin \mathcal{I}$ and $\varepsilon = \operatorname{sgn}(\tau - c) = \pm 1$ for all $\tau \in \mathcal{I}$, an integer $m \geq 1$, a Kähler manifold (N, h) of complex dimension $m-1$, also assumed to be Einstein unless $m = 2$, a C^∞ complex line bundle $\mathcal{L}$ over N, and the horizontal distribution $\mathcal{H}$ of a connection in $\mathcal{L}$ making a fixed Hermitian fibre metric $\langle\,,\rangle$ parallel and having the curvature form $\varOmega = -2\varepsilon a\omega^{(h)}$, where $\omega^{(h)}$ is the Kähler form of (N, h). We also set $r_- = \inf r$ and $r_+ = \sup r$ on $\mathcal{I}$.

We allow here the possibility that $m = 1$, so that N consists of a single point y, and the total space $\mathcal{L} = \{y\} \times \mathcal{L}_y$ may be identified with the fibre $\mathcal{L}_y$. Then, by definition: $\varepsilon = 0$, the "zero metric" h is Einstein, and $\varOmega = 0$.

Let U be the open subset of $\mathcal{L} \smallsetminus N$ given by $r_- < r < r_+$, where, this time, $r : \mathcal{L} \to [0, \infty)$ is the norm function of $\langle\,,\rangle$. We define a metric g on U by

$$(4.1)\quad \begin{array}{ll} \text{i)} & g = f\pi^*h \text{ on } \mathcal{H}, \quad g = (ar)^{-2} Q \operatorname{Re}\langle\,,\rangle \text{ on } \mathcal{V}, \quad g(\mathcal{H}, \mathcal{V}) = \{0\}, \quad \text{where} \\ \text{ii)} & f = 1 \quad (\text{when } \varepsilon = 0), \quad \text{or} \quad f = 2|\tau - c| \quad (\text{when } \varepsilon = \pm 1). \end{array}$$

The symbols $\pi, \mathcal{V}, \mathcal{H}$ and $\langle\,,\rangle$ stand here for the bundle projection $\mathcal{L} \to N$, the vertical and horizontal distributions, and the fixed Hermitian fibre metric in $\mathcal{L}$, while $\operatorname{Re}\langle\,,\rangle$ is the standard Euclidean metric on each fibre of $\mathcal{L}$, and the inverse diffeomorphism $r \mapsto \tau$ of $\tau \mapsto r$ is used to treat functions of $\tau \in \mathcal{I}$ as functions of $r \in (r_-, r_+)$, so that r, τ, Q, f now become C^∞ functions $U \to \mathbf{R}$. The last relation in (4.1.i) means that $\mathcal{H}$ is g-orthogonal to $\mathcal{V}$.

According to Remark 2.4, $\mathcal{L}$ has a unique structure of a holomorphic line bundle over N such that $\mathcal{H}$ is J-invariant. This turns $U \subset \mathcal{L}$ into a complex manifold with $\dim_{\mathbf{C}} U = m$. As shown in [**7**], §16 (especially Remark 16.1),

(a) g is a Kähler metric on U, for which τ is a special Kähler-Ricci potential,
(b) Q treated as a function $U \to \mathbf{R}$ is given by $Q = |\nabla\tau|^2$.

Remark 4.1. Whenever $\mathcal{I}, \tau, Q, r, a, \varepsilon, c, m, N, h, \mathcal{L}, \mathcal{H}, \langle\,,\rangle$ satisfy the above assumptions, the same assumptions hold for the new set of data obtained by leaving $\mathcal{I}, \tau, Q, \varepsilon, c, m, N, h$ unchanged, and replacing the function r of the variable τ by $r^* = 1/r$, the constant a by the constant $a^* = -a$, and $\mathcal{L}, \langle\,,\rangle, \mathcal{H}$ by the dual complex line bundle $\mathcal{L}^*$ with the corresponding dual objects $\langle\,,\rangle^*, \mathcal{H}^*$.

In fact, the curvature forms $\varOmega, \varOmega^*$ of a given connection and its dual differ only by sign, since so do their connection forms $\varGamma, \varGamma^*$ (see (2.1.vii)) relative to two local sections, without zeros, that have the form w and w^{-1} (cf. Remark 2.5).

Remark 4.2. Let φ be a C^{k+1} function, $0 \le k \le \infty$, of a real variable s, defined on an interval containing 0 (possibly as an endpoint), and such that $\varphi(0) = 0$. Then $\varphi(s)/s$ can be extended to a C^k function of s defined on the same interval, including $s = 0$. In fact, integrating $d[\varphi(s\sigma)]/d\sigma$ we obtain the Taylor formula $\varphi(s) = sH(s)$, where $H(s) = \int_0^1 \dot\varphi(s\sigma)\,d\sigma$ with $\dot\varphi = d\varphi/ds$.

Remark 4.3. If Q is a C^∞ function of the real variable τ, defined on a half-open interval $\mathcal{I}'$, positive on its interior $\mathcal{I}$, and such that $Q = 0$ and $dQ/d\tau = 2a \ne 0$ at the only endpoint τ_0 of $\mathcal{I}'$, then, for any positive C^∞ function r of $\tau \in \mathcal{I}$ with $dr/d\tau = ar/Q$, setting $r_+ = \sup r$ on $\mathcal{I}$, we have

- (i) $r \to 0$ as $\tau \to \tau_0$, while Q/r^2 has a positive limit as $\tau \to \tau_0$,
- (ii) τ and Q/r^2 are C^∞ functions of $r^2 \in [0, r_+^2)$ with $Q/r^2 > 0$ at $r = 0$.

In fact, $Q/(\tau - \tau_0)$ is a C^∞ function of $\tau \in \mathcal{I}'$ equal to $2a$ at $\tau = \tau_0$ (Remark 4.2), and so $2d[\log r]/d\tau = 2a/Q$ equals $1/(\tau - \tau_0)$ plus a C^∞ function of τ, that is, $\log r^2$ equals $\log|\tau - \tau_0|$ plus a C^∞ function of $\tau \in \mathcal{I}'$. Hence *$r^2/(\tau - \tau_0)$ is a C^∞ function of $\tau \in \mathcal{I}'$ with a nonzero value at τ_0.* Now $Q/|\tau - \tau_0|$ and $|\tau - \tau_0|/r^2$ both have positive limits as $\tau \to \tau_0$ (the former limit being $2|a|$), and so the same follows for Q/r^2, which proves (i). In view of the statement italicized above, the assignment $\tau \mapsto r^2$ is a C^∞ diffeomorphism of $\mathcal{I}'$ onto $[0, r_+^2)$, sending the endpoint τ_0 to 0, and so (i) implies (ii).

Part (ii) of the following lemma shows how the construction described above can be modified, so as to yield a special Kähler-Ricci potential τ on a Kähler manifold (U^{o}, g) with a critical manifold of complex codimension one (namely, the zero section N). The examples thus obtained constitute local models for one of the two possible cases of a local classification, similar to that mentioned at the beginning of this section, but this time valid at critical points of τ. See Remark 16.4.

Lemma 4.4. *For any data $\mathcal{I}, \tau, Q, r, a, \varepsilon, c, m, N, h, \mathcal{L}, \mathcal{H}, \langle\,,\rangle$ satisfying the assumptions listed above, and the corresponding objects U, g and $\tau : U \to \mathbf{R}$,*

- (i) *the inversion biholomorphism $\mathcal{L}\smallsetminus N \to \mathcal{L}^*\smallsetminus N^*$, described in* Remark 2.5, *sends U, g, τ onto the analogous objects U^*, g^*, τ^* obtained by applying the above construction to the new data introduced in* Remark 4.1;
- (ii) *if, in addition, $\mathcal{I}$ has a finite endpoint τ_0 such that Q admits a C^∞ extension to $\mathcal{I}' = \mathcal{I} \cup \{\tau_0\}$ with $Q = 0$ and $dQ/d\tau = 2a$ at $\tau = \tau_0$, while either $\varepsilon = \pm 1$ and $\tau_0 \ne c$, or $\varepsilon = 0$ and c is undefined, then g and τ have C^∞ extensions to a metric and a function on the open set $U^{\mathrm{o}} \subset \mathcal{L}$ given by $0 \le r < r_+$, that is, on the bundle of radius r_+ open disks in $\mathcal{L}$.*

Proof. Let r, r^* also stand for the norm functions $\mathcal{L} \to \mathbf{R}$ and $\mathcal{L}^* \to \mathbf{R}$ of $\langle\,,\rangle$ and $\langle\,,\rangle^*$. That the inversion biholomorphism sends U onto U^* is clear as U, U^* are given by $r_- < r < r_+$ and $r_-^* < r^* < r_+^*$, with $r_\pm^* = 1/r_\mp$. Next, using the multiplicative notation $\zeta z \in \mathbf{C}$ for evaluating a functional $\zeta \in \mathcal{L}_y^*$ on $z \in \mathcal{L}_y$, we see that the differential at any $(y, z) \in U$ of the inversion biholomorphism acts on vertical vectors $\dot z \in T_{(y,z)}\mathcal{L}_y = \mathcal{V}_y$ via $\dot z \mapsto -(\zeta\dot z)\zeta$, where $\zeta = z^{-1}$, and so it pulls back the Euclidean metric $\mathrm{Re}\langle\,,\rangle^*$ on $T_{(y,\zeta)}\mathcal{L}_y^*$ onto $1/r^4$ times the Euclidean metric on $T_{(y,z)}\mathcal{L}_y$. Hence it sends the restriction of g to the vertical distribution $\mathcal{V}$ in U onto the analogous restriction of g^*, and (i) follows from (4.1) and Remark 2.5. Next, under the assumptions of (ii), the real fibre metric g on $TU = \mathcal{H} \oplus \mathcal{V}$ is the direct sum of $f\pi^*h$ in $\mathcal{H}$ and $\theta\,\mathrm{Re}\langle\,,\rangle$ in $\mathcal{V}$, with $\theta = Q/(ar)^2$ and f as in (4.1.ii). The required extensions exist since the distributions $\mathcal{H}, \mathcal{V}$ and the fibre metrics π^*h and $\mathrm{Re}\langle\,,\rangle$ on them are defined and of class C^∞ everywhere in $\mathcal{L}$, while, by Remark 4.3(ii), the functions τ, θ, f have C^∞ extensions to U^{o}, which are positive in the case of θ and f (the latter due to our assumption that $\tau_0 \neq c$ unless $\varepsilon = 0$). This completes the proof. ∎

§5. Class 1: special Kähler-Ricci potentials on $\mathbf{C}\mathrm{P}^1$ bundles

In this section we construct examples of special Kähler-Ricci potentials τ on compact Kähler manifolds (M, g), in any complex dimension $m \geq 1$. The resulting class of triples (M, g, τ) will be called Class 1. First, let us suppose that

(5.1) $[\tau_{\min}, \tau_{\max}]$ is a nontrivial closed interval of the variable τ with a C^∞ function $[\tau_{\min}, \tau_{\max}] \ni \tau \mapsto Q \in \mathbf{R}$, which is positive on the open interval $(\tau_{\min}, \tau_{\max})$ and vanishes at the endpoints $\tau_{\min}, \tau_{\max}$, while the values of $dQ/d\tau$ at the endpoints are mutually opposite and nonzero.

We then select an endpoint τ_0 of the interval $[\tau_{\min}, \tau_{\max}]$, a C^∞ diffeomorphism $(\tau_{\min}, \tau_{\max}) \ni \tau \mapsto r \in (0, \infty)$ with $dr/d\tau = ar/Q$, where $a \in \mathbf{R}$ is characterized by $dQ/d\tau = 2a$ at $\tau = \tau_0$, and real numbers ε, c such that either $\varepsilon = 0$ (and c is undefined), or $\varepsilon = \mathrm{sgn}\,(\tau - c) = \pm 1$ for all $\tau \in [\tau_{\min}, \tau_{\max}]$. (Cf. Remark 5.1.)

We also fix an integer $m \geq 1$, a compact Kähler manifold (N, h) with $\dim_{\mathbf{C}} N = m - 1$ which is Einstein unless $m = 2$, and a C^∞ complex line bundle $\mathcal{L}$ over N with a $\mathrm{U}(1)$-connection having the curvature form $\Omega = -2\varepsilon a \omega^{(h)}$, where $\omega^{(h)}$ is the Kähler form of (N, h). About the case $m = 1$, see the third paragraph of §4.

Formula (4.1) now defines a Kähler metric g on $U = \mathcal{L} \smallsetminus N$, while τ becomes a special Kähler-Ricci potential on (U, g), when treated, with the aid of the inverse diffeomorphism $r \mapsto \tau$, as a function of the norm function $r : U \to (0, \infty)$. This is immediate from (a) in §4.

Let M denote the projective compactification of $\mathcal{L}$, that is, the holomorphic $\mathbf{C}\mathrm{P}^1$ bundle over N obtained from the disjoint union $\mathcal{L} \cup \mathcal{L}^*$ by identifying the complements of the zero sections $N \subset \mathcal{L}$ and $N^* \subset \mathcal{L}^*$ via the inversion biholomorphism $\mathcal{L} \smallsetminus N \to \mathcal{L}^* \smallsetminus N^*$ described in Remark 2.5.

Our Class 1 triple (M, g, τ) arises since both g and τ have C^∞ extensions to a Kähler metric and a special Kähler-Ricci potential on M, again denoted by g, τ.

In fact, by (5.1), the additional assumptions in Lemma 4.4(ii) hold both for the original data and for the "dual" ones, obtained when $r^* = 1/r$, $a^* = -a$, $\mathcal{L}^*$ and the other endpoint are used instead of $r, a, \mathcal{L}$ and τ_0, while the connection and fibre metric in $\mathcal{L}$ are replaced by their duals in $\mathcal{L}^*$, and the other ingredients are left unchanged. Now, in view of Lemma 4.4(i), the inversion biholomorphism sends our pair g, τ on $\mathcal{L} \smallsetminus N$ onto analogous objects on $\mathcal{L}^* \smallsetminus N^*$, obtained as above from the dual data, while, by Lemma 4.4(ii), both pairs admit extensions to $\mathcal{L}$ and, respectively, $\mathcal{L}^*$.

Remark 5.1. That $\tau \mapsto r$ maps $(\tau_{\min}, \tau_{\max})$ *onto* $(0, \infty)$ is clear as $Q = 0 \neq dQ/d\tau$ at $\tau_{\min}$ and $\tau_{\max}$, and so the limits of $\log\, r$ at both endpoints are infinite.

Remark 5.2. For (M,g) and τ obtained above, τ has two critical manifolds: the zero sections $N \subset \mathcal{L}$ and $N^* \subset \mathcal{L}^*$. This is clear from (b) in §4, since $Q = 0$ only at the endpoints of $[\tau_{\min}, \tau_{\max}]$, which correspond to $r = 0$ and $r = \infty$.

Remark 5.3. Some of the Class 1 triples (M,g,τ) constructed above turn out to be (gradient) Kähler-Ricci solitons, namely, special cases of examples found by Koiso [**14**] (for a brief exposition, see [**19**], §4).

§6. Class 2: special Kähler-Ricci potentials on $\mathbf{C}\mathrm{P}^m$

We will now describe examples of special Kähler-Ricci potentials τ on compact Kähler manifolds (M,g), in any complex dimension $m \geq 1$, such that M is biholomorphic to $\mathbf{C}\mathrm{P}^m$. The triples (M,g,τ) constructed here form what we refer to as Class 2. A different description of Class 2 was given earlier, in Example 3.6 (see Remark 6.2); the presentation in this section is, however, much more explicit, thus providing not only a clear picture of the corresponding moduli space, but also a convenient reference for the classification argument in §16.

A part of the construction described below works under assumptions weaker than those made here; the resulting examples of special Kähler-Ricci potentials with isolated critical points on noncompact Kähler manifolds play a crucial role in a local classification result which generalizes the one mentioned at the beginning of §4. See Remark 16.4.

As in the construction of §5 that led to Class 1, we fix $\tau_{\min}, \tau_{\max}$ and Q satisfying the positivity-and-boundary conditions (5.1). Then we choose an endpoint c of $[\tau_{\min}, \tau_{\max}]$, a C^∞ diffeomorphism $(\tau_{\min}, \tau_{\max}) \ni \tau \mapsto r \in (0,\infty)$ with $dr/d\tau = ar/Q$ for $a \in \mathbf{R} \smallsetminus \{0\}$ such that $2a = dQ/d\tau$ at $\tau = c$, a complex vector space V with $\dim_{\mathbf{C}} V = m \geq 1$, and a Hermitian inner product $\langle\,,\rangle$ in V. (The image of $\tau \mapsto r$ is $(0,\infty)$, cf. Remark 5.1; also, $r \to 0$ as $\tau \to c$, due to the choice of a.) As for the case $m = 1$, see the paragraph preceding (4.1).

We now define a Riemannian metric g on $V \smallsetminus \{0\}$ by $|a|r^2 g = 2|\tau - c|\,\mathrm{Re}\langle\,,\rangle$ on $\mathcal{H}$, $a^2 r^2 g = Q\,\mathrm{Re}\langle\,,\rangle$ on $\mathcal{V}$ and $g(\mathcal{H},\mathcal{V}) = \{0\}$, where $\mathcal{V}$ is the distribution on $V \smallsetminus \{0\}$ with $\mathcal{V}_x = \mathbf{C}x$ and $\mathcal{H}$ is its orthogonal complement relative to the Euclidean metric $\mathrm{Re}\langle\,,\rangle$, while r also stands for the norm function $V \to [0,\infty)$, and the inverse diffeomorphism $r \mapsto \tau$ of $\tau \mapsto r$ is used to treat τ and Q as functions $V \to \mathbf{R}$.

Finally, let M be the projective space, biholomorphic to $\mathbf{C}\mathrm{P}^m$, of all complex lines through 0 in $V \times \mathbf{C}$. We treat V as an open subset of M using the standard holomorphic embedding $V \ni x \mapsto \mathrm{Span}_{\mathbf{C}}\{(x,1)\} \in M$. Both g and τ then have C^∞ extensions to M, again denoted by g, τ, such that g is a Kähler metric on M and τ is a special Kähler-Ricci potential on (M,g).

We prove the claim made in the last sentence using three steps. First, we will verify that g is a Kähler metric on $V \smallsetminus \{0\}$ and τ is a special Kähler-Ricci potential on $(V \smallsetminus \{0\}, g)$. In the second (or, third) step we will show the existence of the required extensions of g and τ from $V \smallsetminus \{0\}$ to $M \smallsetminus \{0\}$ (or, respectively, to V).

The first step is immediate from conclusions (a), (b) in §4, since our definition of g is a special case of (4.1) for the data $\mathcal{I}, \tau, Q, r, a, \varepsilon, c, m, N, h, \mathcal{L}, \mathcal{H}, \langle\,,\rangle$ formed by $\mathcal{I} = (\tau_{\min}, \tau_{\max})$ with $\tau \mapsto r$ and Q, a, c, m chosen above, $\varepsilon = \pm 1$ with $\varepsilon a > 0$ (so that $\varepsilon(\tau - c) > 0$ for all $\tau \in \mathcal{I}$, due to our choice of a), $N \approx \mathbf{C}\mathrm{P}^{m-1}$ defined to be the projective space of V, the metric h on N equal to $1/|a|$ times the Fubini-Study metric g_{FS}, the tautological line bundle $\mathcal{L}$ over N, as well as $\mathcal{H} = \mathcal{V}^{\perp}$ and $\langle\,,\rangle$ appearing earlier in this section. More precisely, $\mathcal{H}$ and $\langle\,,\rangle$ make sense as objects in $\mathcal{L}$ due to the standard biholomorphic identification $\mathcal{L} \smallsetminus N = V \smallsetminus \{0\}$ given, in the notation of (2.1.v), by $(y, z) \mapsto z$. The restriction to the fibres of $\mathcal{L}$ of the inner product $\langle\,,\rangle$ in V then is a fibre metric in $\mathcal{L}$, while $\mathcal{H} = \mathcal{V}^{\perp}$ is the horizontal distribution of the connection in $\mathcal{L}$ obtained by projecting the standard flat connection in the product bundle $\mathcal{E} = N \times V$ onto the $\mathcal{L}$ summand in the decomposition $\mathcal{E} = \mathcal{L} \oplus \mathcal{L}^{\perp}$. Finally, $\varOmega = -2\varepsilon a\omega^{(h)}$, as required in §4, that is, $\varOmega = -2\omega_{\mathrm{FS}}$ for the Kähler form ω_{FS} of g_{FS}. Namely, $\varOmega$ and ω_{FS} are invariant under the action of the unitary group of V on $N \approx \mathbf{C}\mathrm{P}^{m-1}$, and so $\varOmega$ equals a constant times ω_{FS}, while integrating over a fixed complex projective line $S \subset N$ we get $\int_S \varOmega = -2\pi = -2\int_S \omega_{\mathrm{FS}}$. (That the complex-manifold structure of $V \smallsetminus \{0\}$ agrees with the holomorphic-bundle structure of $\mathcal{L}$ provided by Remark 2.4 is clear from uniqueness of the latter.)

In the second step, let $\mathcal{L}^*$ be the dual of the tautological bundle $\mathcal{L}$ over N. Using the notation of (2.1.v), we define a mapping $\mathcal{L}^* \to M \smallsetminus \{0\}$ by assigning to (y, ζ) the graph of the linear functional $\zeta \in y^*$. (The graph is an element of M, as it is a line in $y \times \mathbf{C} \subset V \times \mathbf{C}$.) This mapping is obviously a biholomorphism; restricted to the complement of the zero section in $\mathcal{L}^*$, it becomes, under our identification $\mathcal{L} \smallsetminus N = V \smallsetminus \{0\}$, the inverse of the inversion biholomorphism of Remark 2.5, and the required conclusion is immediate from Lemma 4.4(ii).

For the third step, let the vector fields v, u and 1-forms ξ, ξ' on V be given by $v(x) = ax$, $u(x) = iax$, for our a, and $\xi = \mathrm{Re}\langle v, \cdot\rangle$, $\xi' = \mathrm{Re}\langle u, \cdot\rangle$. Then g on $V \smallsetminus \{0\}$ is the combination of $\xi \otimes \xi + \xi' \otimes \xi'$ and $\mathrm{Re}\langle\,,\rangle$ with the coefficients $[Q - 2a(\tau - c)]/(ar)^4$ and $2(\tau - c)/(ar^2)$, where $(\xi \otimes \xi')(w, w') = \xi(w)\xi'(w')$ for tangent vectors w, w'. In fact, as $(\tau - c)/a > 0$, both g and this combination yield the same value when evaluated on two vectors, of which one is in $\mathcal{H}$ and the other in $\mathcal{H}$ or $\mathcal{V}$ and, similarly, the same value when evaluated on v, v, or v, u, or u, u (cf. the line preceding Remark 2.4). Next, both coefficients are C^∞ functions of the variable $r^2 \in [0, \infty)$. Namely, for $2(\tau - c)/(ar^2)$ this is clear from Remarks 4.2 (with $s = r^2$) and 4.3(ii) (with $\tau_0 = c$). Next, Q/r^2 and $2a(\tau - c)/r^2$, treated as C^∞ functions of $r^2 \in [0, r_+^2)$ (see Remark 4.3), have the same positive value at $r^2 = 0$, since $dr/d\tau = ar/Q$ and so $Q/r^2 = 2a\,d\tau/d(r^2)$. Thus, their difference divided by r^2 is a C^∞ function of $r^2 \in [0, r_+^2)$ (from Remark 4.2 for $s = r^2$). Positivity of $2a(\tau - c)/r^2$ at $r^2 = 0$ also shows that the limit of g at $0 \in V$ is positive definite, completing the third step of our argument.

Remark 6.1. For (M, g) and τ constructed as above, τ has two critical manifolds: the one-point set $\{0\} \subset V \subset M$, and $M \smallsetminus V$ (the hyperplane at infinity), due to (b) in §4 and the fact that $Q > 0$ on $(\tau_{\min}, \tau_{\max})$, while $Q = 0$ at the endpoints of $[\tau_{\min}, \tau_{\max}]$, which correspond to $r = 0$ and $r = \infty$ (cf. Remark 5.1).

Remark 6.2. All triples (M, g, τ) constructed here obviously have the properties listed in Example 3.6. Conversely, every triple $(\mathbf{C}\mathrm{P}^m, g, \tau)$ of Example 3.6 can also be obtained as described in this section. This fact, which will not be used, is an immediate consequence of Theorem 16.3.

§7. Dimensions of critical manifolds

Now that we have described two classes of examples, we proceed to the second part of our presentation; it will culminate in Theorem 16.3, classifying all special Kähler-Ricci potentials τ on compact Kähler manifolds. This section is a first major step towards the proof of Theorem 16.3. (The next such step is §11.) Specifically, Proposition 7.3 states that the complex dimension of a critical manifold N of τ must be 0 or $\dim_{\mathbf{C}} M - 1$, and describes the structure of the Hessian of τ along N.

Remark 7.1. Given a special Kähler-Ricci potential τ on a Kähler manifold (M, g), let $Q = g(v, v)$ for $v = \nabla\tau$, and let ϕ, ψ be as in (3.1).

(i) Writing $\dot{f} = d[f(x(s))]/ds$ for a fixed C^1 curve $s \mapsto x(s) \in M$ and a C^1 function f defined in M, we have $\dot{f} = d_{\dot{x}} f = g(\nabla f, \dot{x})$, where $\dot{x} = dx/ds$. Consequently, $g(v, \dot{x}) = \dot{\tau}$ and, by (3.2.b), $\dot{Q} = 2\psi\dot{\tau}$.

(ii) Given $y \in M$ with $v(y) = 0$, let $s \mapsto x(s) \in M$ be a C^2 curve such that $x(0) = y$ and $\dot{x}(0)$ is an eigenvector of $(\nabla d\tau)(y)$ for an eigenvalue $a \neq 0$ (which exists since $\nabla d\tau \neq 0$ at y by Remark 2.3(i).) Then $\dot{\tau} = 0$ and $\ddot{\tau} = a|\dot{x}|^2 \neq 0$ at $s = 0$ (notation of (i)), so that $\dot{\tau} \neq 0$ for all $s \neq 0$ near 0. In fact, $\dot{\tau} = g(v, \dot{x})$ (see (i)), and so $\ddot{\tau} = g(\nabla_{\dot{x}} v, \dot{x}) + g(v, \nabla_{\dot{x}} \dot{x})$, which at $s = 0$ equals $(\nabla d\tau)(\dot{x}, \dot{x})$ (by (2.1.iii), as $v(y) = 0$).

In the next lemma, critical manifolds are defined as in §2, and we use the notation of §3, for a special Kähler-Ricci potential τ on a Kähler manifold: M' is the set of non-critical points for τ, and $\phi, \psi : M' \to \mathbf{R}$ are characterized by (3.1). Thus, by Lemma 3.1(i), either $\phi = 0$ identically or $\phi \neq 0$ everywhere in M'. Next, c is the constant defined (only when $\phi \neq 0$ on M') in Lemma 3.1(ii). Finally, a $2m$-tuple such as $\phi(y), \ldots, \phi(y), \psi(y), \psi(y)$ stands for $\psi(y), \psi(y)$ when $m = 1$.

Lemma 7.2. *Let τ be a special Kähler-Ricci potential on a Kähler manifold (M, g). The functions ϕ and ψ then have unique continuous extensions to M, which we also denote by ϕ and ψ. Next, the eigenvalues of $\nabla d\tau$ at any critical point y of τ, listed with their multiplicities, are $\phi(y), \ldots, \phi(y), \psi(y), \psi(y)$, and the eigenspace of $\nabla d\tau$ for the eigenvalue $\psi(y)$ is the normal space, at y, of the critical manifold of τ containing y. Finally, for any critical point y of τ,*

(I) *$\phi(y) = 0 \neq \psi(y)$ if either $\phi = 0$ on M', or $\phi \neq 0$ on M' and $\tau(y) \neq c$,*

(II) *$\psi(y) = \phi(y) \neq 0$, if $\phi \neq 0$ on M' and $\tau(y) = c$.*

Proof. Let y be a critical point of τ. Since M' is dense in M (Remark 2.3(ii)), we may choose a sequence of points in M' converging to y, but otherwise arbitrary, and select, at each point x of the sequence, an orthonormal basis of $T_x M$ formed by eigenvectors of $(\nabla d\tau)(x)$ ordered so that the corresponding eigenvalues are $\phi(x), \ldots, \phi(x), \psi(x), \psi(x)$ (cf. (3.1)). A subsequence of the sequence of bases converges, in a suitable frame bundle, to an orthonormal basis of $T_y M$ consisting of eigenvectors of $(\nabla d\tau)(y)$ for some eigenvalues $\phi_0, \ldots, \phi_0, \psi_0, \psi_0$ such that ϕ_0 and ψ_0 are the limits of the sequences $\phi(x)$ and $\psi(x)$.

Considering two separate cases ($\psi_0 = \phi_0$ and $\psi_0 \neq \phi_0$) we easily see that the limits ϕ_0 and ψ_0 do not depend on the choice of a sequence in M' converging to y. Hence $\phi(x) \to \phi_0$ and $\psi(x) \to \psi_0$ as $x \to y$, where x is a variable point of M'.

We thus obtain the required continuous extensions, with $\phi(y) = \phi_0$ and $\psi(y) = \psi_0$. The assertion about the eigenvalues is now obvious as well.

Relation $\psi(y) \neq 0$ in (I), (II) will clearly follow from the remainder of (I) and (II), since, as we saw, $\phi(y), \ldots, \phi(y), \psi(y), \psi(y)$ form the spectrum of $(\nabla d\tau)(y)$, while $(\nabla d\tau)(y) \neq 0$ (Remark 2.3(i)). To prove (I), we may now assume that $\phi \neq 0$ on M' and $\tau(y) \neq c$, and then let $x \to y$, for $x \in M'$, in the equality $|\nabla\tau|^2 = Q = 2(\tau - c)\phi$ (Lemma 3.1(ii)). For (II), let us choose a curve $s \mapsto x(s)$ as in Remark 7.1(ii), so that $\dot\tau \neq 0$ for all $s \neq 0$ close to 0. Hence, by l'Hospital's rule, $Q/(\tau - c)$ evaluated at $x(s)$ tends, as $s \to 0$, to the limit of $\dot Q/(\tau - c)^{\cdot} = 2\psi\dot\tau/\dot\tau$ (see Remark 7.1(i)), that is, to $2\psi(y)$, while, by Lemma 3.1(ii), $Q/(\tau - c) \to 2\phi(y)$.

The conclusion about the eigenspace is also immediate, as the tangent space at y of the critical manifold containing y is the nullspace of $(\nabla d\tau)(y)$ (cf. (d) in Remark 2.3(iii) and (2.1.iii)). This completes the proof. ■

Continuity of the extensions in Lemma 7.2 can actually be replaced by their C^∞ differentiability, which we will not use; cf. Lemma 9.1 and (3.2.c).

The following proposition establishes a crucial dichotomy involving the dimensions of critical manifolds of special Kähler-Ricci potentials. Namely, there are just two possible cases, corresponding to (I) and (II) in Lemma 7.2.

Proposition 7.3. *Every critical manifold N of a special Kähler-Ricci potential τ on a Kähler manifold (M, g) is a complex submanifold of complex codimension one, or consists of a single point.*

Furthermore, there exists a real constant a depending on N which, at any $y \in N$, is the unique nonzero eigenvalue of $\nabla d\tau$.

Proof. If $y \in N$, the spectrum of $(\nabla d\tau)(y)$ is $\{0, \ldots, 0, a, a\}$ in case (I) (of Lemma 7.2), or $\{a, \ldots, a\}$ in case (II), with $a = \psi(y)$, while, by Lemma 7.2, $(T_yN)^\perp$ is its eigenspace for the unique nonzero eigenvalue a. Thus, the first assertion follows. To obtain constancy of a on N (that is, its independence of $y \in N$), we may assume that $\dim_{\mathbf{C}} M = m \geq 2$. One of conditions (I), (II) in Lemma 7.2 now must hold for *all* $y \in N$, as the choice between (I) and (II) is determined by $\dim_{\mathbf{C}} N$. It now follows that a is constant on N. Namely, by the above description of the spectrum of $(\nabla d\tau)(y)$, the value of $Y = \Delta\tau$ at y is $2a$ in case (I) and $2ma$ in case (II). However, $dY = -2\mathrm{r}(\nabla\tau, \cdot)$ (Remark 2.1), so that $dY = 0$ wherever $d\tau = 0$, and hence Y is constant on N. This completes the proof. ■

The final clause of Proposition 7.3 provides a detailed description of the structure of the Hessian $\nabla d\tau$ at any point $y \in N$. Namely, if a is the only eigenvalue, $\nabla d\tau$ equals ag at y, while, if it is not, the other eigenvalue must be 0, with the eigenspace T_yN (see (d) in Remark 2.3(iii)). In both cases, the a-eigenspace of $\nabla d\tau$ at y is the normal space $(T_yN)^\perp$. Thus, with $v = \nabla\tau$ and $u = Jv$, we have,

$$\begin{aligned} &\nabla_w v = \nabla_w u = 0 \text{ if } w \in T_yN, \text{ and} \\ &\nabla_w v = aw,\ \nabla_w u = aJw \text{ if } w \in (T_yN)^\perp. \end{aligned} \tag{7.1}$$

For v this is now clear since ∇v has the same eigenvalues and eigenvectors at y as $\nabla d\tau$ (cf. (2.1.iii)); for u, it in turn follows since $\nabla J = 0$.

Remark 7.4. Suppose that τ is a special Kähler-Ricci potential on a Kähler manifold (M,g). The continuous extensions $\phi,\psi: M \to \mathbf{R}$ of Lemma 7.2 then are constant along any critical manifold N of τ. More precisely, in terms of the constant $a \neq 0$ associated with N as in Proposition 7.3, $\phi = 0$ on N and $\psi = a$ on N if $\dim_{\mathbf{C}} N = \dim_{\mathbf{C}} M - 1$, while $\phi = \psi = a$ on N if N is a one-point set.

This is clear as $\psi = a$ on N, cf. Proposition 7.3 and Lemma 7.2.

Lemma 7.5. *Let τ be a special Kähler-Ricci potential on a Kähler manifold (M,g) of complex dimension $m \geq 2$, and let c and ε be as in* Lemma 3.1.

(i) *If $\varepsilon = 0$, no critical manifold of τ is a one-point set.*
(ii) *If $\varepsilon \neq 0$, a point $y \in M$ satisfies the condition $\tau(y) = c$ if and only if $\{y\}$ is a critical manifold of τ.*

In fact, (i) is obvious from Remark 7.4 (as condition $\phi = 0$ on M' contradicts $\phi = \psi = a \neq 0$ on N). Now let $y \in M$. Assertion (ii) is obvious when $d\tau \neq 0$ at y (namely, one then has $\tau(y) \neq c$, as $|\nabla\tau|^2 = Q = 2(\tau - c)\phi$ by Lemma 3.1). Finally, let y be a critical point of τ, and let N be the critical manifold of τ containing y. Using Lemma 7.2, we see that if $\tau(y) \neq c$ (or, $\tau(y) = c$), the complex dimension of $(T_y N)^\perp$ equals 1 (or, respectively, m), which yields (ii) in this case as well. ■

§8. Geodesic vector fields

The main result of this section is Lemma 8.3, stating that the gradient of a special Kähler-Ricci potential τ on a Kähler manifold is tangent to all sufficiently short normal geodesic segments emanating from any critical manifold of τ. This fact will be used in §9 to obtain a differentiability assertion (Lemma 9.1), needed for our classification result (Theorem 16.3).

Let ∇ be a fixed connection in the tangent bundle TM of a manifold M. (It need not be the Levi-Civita connection of a Riemannian metric.) Condition

(i) $\nabla_v v = \psi v$ for some function $\psi: M \to \mathbf{R}$

imposed on a C^∞ vector field v on M, implies C^∞-differentiability of ψ on the open set where $v \neq 0$ (but not necessarily continuity of ψ on M). Moreover,

(ii) a C^∞ vector field v on M satisfies (i) if and only if all its integral curves $s \mapsto x(s)$ are (re-parameterized) geodesics of ∇,
(iii) if (i) holds for a C^∞ vector field v on M and $X \subset M$ is a geodesic segment such that $v(x)$ is tangent to X at some point $x \in X$, while $v \neq 0$ at all points of X, then v is tangent to X at every point of X.

In fact, for integral curves, $\dot{x}(s) = v(x(s))$ (with $\dot{x} = dx/ds$), and so $\nabla_{\dot{x}}\dot{x}$ at any s equals $\nabla_v v$ at $x(s)$. Now (ii) follows since curves $s \mapsto x(s)$ obtained by re-parameterizing geodesics are characterized by $\nabla_{\dot{x}}\dot{x} = \psi\dot{x}$, where ψ, this time, is a function of s. To verify (iii), note that both X and the underlying set $\tilde{X}$ of the maximal integral curve of v containing x are geodesics (by (ii)), tangent to each other at x, and so $x \in X' \subset \tilde{X}$ for some nontrivial subsegment X' of X, which may be chosen maximal with this property. Then $X' = X$, for otherwise an endpoint x' of X' would be an interior point of X and $v(x') \neq 0$ would be tangent to X at x', thus allowing X' to be extended past x' despite its maximality.

Remark 8.1. Given a connection ∇ in the tangent bundle TM, we use the standard symbol $\exp_x: U_x \to M$ for the geodesic exponential mapping of ∇ at any point $x \in M$. Here U_x is a neighborhood of the zero vector in T_xM, namely, the union of maximal line segments emanating from zero on which $\exp_x$ is defined. Thus, $s \mapsto x(s) = \exp_x sw$ is the geodesic with $x(0) = x$ and $\dot{x}(0) = w \in T_xM$. A related mapping is $\mathrm{Exp} : U^{\mathrm{Exp}} \to M$ with $\mathrm{Exp}(x, w) = \exp_x w$, defined on the set $U^{\mathrm{Exp}} = \bigcup_{x \in M}(\{x\} \times U_x)$ in the total space $TM = \{(x, w) : x \in M,\ w \in T_xM\}$, containing the zero section M (cf. (2.1.v)). It is well known that U^{Exp} is open in TM and Exp is of class C^∞. (See [**12**], Proposition 8.1 in Ch. III.)

Lemma 8.2. *Suppose that ∇ is a connection in the tangent bundle TM of a manifold M and v is a C^∞ vector field on M with* (i), *while $X' \subset M$ is a geodesic segment containing an endpoint y with $v(y) = 0$. If $\nabla_w v = aw \neq 0$ for some vector w tangent to X' at y and some $a \in \mathbf{R}$, then*

(a) *there exists a nontrivial compact subsegment X of X', containing y, and such that $v(x) \neq 0$ for all $x \in X \smallsetminus \{y\}$,*
(b) *for any subsegment $X \subset X'$ with the properties listed in* (a) *we have $v(x) \in T_xX$ at every $x \in X$.*

Proof. Let $s \mapsto x(s)$ be a geodesic parameterization of X' with $x(0) = y$, defined on a subinterval of $[0, \infty)$. Thus, $\nabla_{\dot{x}}\dot{x} = 0$, where $\dot{x} = dx/ds$. A fixed 1-form ξ of class C^∞ on a neighborhood U of y such that $a\xi(w) > 0$ at y, for $w = \dot{x}(0)$, gives rise to a C^∞ function $\varphi = \xi(v) : U \to \mathbf{R}$ with $\varphi = 0$ wherever $v = 0$ in U and $d[\varphi(x(s))]/ds > 0$ for all $s \geq 0$ near 0 (as $d_w\varphi = a\xi(w)$), which proves (a).

For X as in (a), let $\ell > 0$ be such that $x(\ell)$ is an endpoint of X, and let $s \mapsto w(s) \in T_{x(s)}M$ be the vector field along X given by $w(0) = \dot{x}(0)$ and $w(s) = v(x(s))/f(s)$ for $s \in (0, \ell]$, where $f : [0, \ell] \to \mathbf{R}$ is any fixed C^1 function with $f(0) = 0$, $\dot{f}(0) = a$ and $|f| > 0$ on $(0, \ell]$. Thus, $w(s) \neq 0$ for all $s \in [0, \ell]$ due to our choice of X and ℓ. Also, setting $\tilde{v}(s) = v(x(s))$ we have $\tilde{v}(0) = 0$, while $\nabla_{\dot{x}}\tilde{v}$ at $s = 0$ equals $\nabla_w v = aw$, with $w = \dot{x}(0)$. This, along with l'Hospital's rule, shows that the mapping $[0, \ell] \ni s \mapsto (x(s), w(s))$, valued in the total space TM (see Remark 8.1), is continuous, also at $s = 0$.

For any fixed $s \in [0, \ell]$, let $r \mapsto x_s(r) \in M$ be the geodesic with $x_s(s) = x(s)$ and $[dx_s(r)/dr]_{r=s} = w(s)$, defined on the maximal possible interval containing s. Then, for any sufficiently small $\varepsilon \in (0, \ell]$,

(A) $r \mapsto x_s(r)$ is defined on an interval containing $[0, \ell]$, for every $s \in [0, \varepsilon]$,
(B) $v \neq 0$ at $x_s(r)$ for any s, r with $0 < s \leq r \leq \varepsilon$.

In fact, if there were no $\varepsilon \in (0, \ell]$ with (A), we could find values of $s \in (0, \ell]$ arbitrarily close to 0 such that one of the points $(x(s), -sw(s))$, $(x(s), (\ell - s)w(s))$ lies in the complement $TM \smallsetminus U^{\mathrm{Exp}}$, with U^{Exp} as in Remark 8.1. Since $TM \smallsetminus U^{\mathrm{Exp}}$ is a closed set, it would then also contain the limit of one of these points as $s \to 0$, that is, $(y, 0)$ or $(y, \ell\dot{x}(0))$, contradicting either the inclusion relation $M \subset U^{\mathrm{Exp}}$, or our choice of ℓ.

Also, $d[\varphi(x_s(r))]/dr > 0$ for all sufficiently small $r, s \in [0, \ell]$ since, due to our choice of φ, this is the case for $r = s = 0$. As $\varphi > 0$ on a nontrivial subsegment of X containing y, except for the point y at which $\varphi = 0$, making $\varepsilon > 0$ with (A) smaller we now get $\varphi > 0$ (and so $v \neq 0$) at $x_s(r)$ for any s, r with $s \in (0, \varepsilon]$ and $s \leq r \leq s + \varepsilon$, proving (B).

By (A), (B) and (iii) above, if $s \in (0, \varepsilon)$, the geodesic $[s, \ell] \ni r \mapsto x_s(r)$ is a (re-parameterized) integral curve of v, and so v is tangent to it at the point $x_s(\varepsilon)$.

Taking the limit as $s \to 0$, we now see that v is tangent to the limiting geodesic X at $x(\varepsilon)$. Hence (iii) for $x = x(\varepsilon)$ and X gives (b), completing the proof. ∎

Lemma 8.3. *Let N be a critical manifold of a special Kähler-Ricci potential τ on a Kähler manifold (M, g). For any unit-speed geodesic $[0, \ell) \ni s \mapsto x(s) \in M$ normal to N at $x(0) = y \in N$ and such that $d\tau \neq 0$ at $x(s)$ whenever $s \in (0, \ell)$, setting $v = \nabla\tau$, $Q = |\nabla\tau|^2$ and $d\tau/ds = d[\tau(x(s))]/ds$, we have, at any $s \in [0, \ell)$,*

(i) $dx/ds = (\operatorname{sgn} a)\, v/|v|$, *if* $s > 0$,
(ii) $d\tau/ds = (\operatorname{sgn} a)\sqrt{Q}$, *with the initial value* $\tau = \tau_0$ *at* $s = 0$,

where τ_0 is the value of τ on N and a depends on N as in Proposition 7.3.

In fact, $\nabla_v v = \psi v$ by (3.2.a) with $w = v$, and $\nabla_w v = aw$ for $w = \dot{x}(0)$ by (7.1), where $\dot{x} = dx/ds$, so that Lemma 8.2(b) for the Levi-Civita connection ∇ of g, the geodesic segment $X \subset M$ which is the image of $s \mapsto x(s)$, and our y, v, a, gives $\dot{x} = \pm v/|v|$ for some sign $\pm$ and all $s \in (0, \ell)$. Next, $\dot{\tau} = g(v, \dot{x})$ and $\dot{Q} = 2\psi\dot{\tau}$ (cf. Remark 7.1(i)). At any $s > 0$ close to 0 we thus have $\pm\dot{\tau} > 0$ and, as $a\psi > 0$ by Remark 7.4, also $\pm a\dot{Q} > 0$. Consequently, $\pm a > 0$, since $\dot{Q} > 0$. (Note that $Q(x(s)) > 0$ for such s, while $Q(x(0)) = 0$.) This yields (i), and (ii) now follows since $d\tau/ds = \dot{\tau} = g(v, dx/ds)$ and $|v| = \sqrt{Q}$. ∎

§9. A differentiability result

In this section we use Gauss's Lemma to show that, for a special Kähler-Ricci potential τ on a Kähler manifold, $Q = |\nabla\tau|^2$ must, locally, be a C^∞ function of τ.

The *normal exponential mapping* of a submanifold N of a Riemannian manifold (M, g) is the restriction of $\operatorname{Exp} : U^{\operatorname{Exp}} \to M$ to the set $U^{\operatorname{Exp}} \cap \mathcal{L}$, where $\mathcal{L}$ is the total space of the normal bundle of N, while $U^{\operatorname{Exp}} \subset TM$ and Exp are defined as in Remark 8.1 for the Levi-Civita connection ∇.

For $M, g, N, \mathcal{L}$ as above, let $s : \mathcal{L} \to [0, \infty)$ be the norm function of the real fibre metric in $\mathcal{L}$ obtained by restricting g to $\mathcal{L}$. For any $y \in N$, the inverse mapping theorem allows us to choose a connected neighborhood N' of y in N and a number $\ell \in (0, \infty)$ such that, for the open subset U' of $\mathcal{L}'$ given by $0 \le s < \ell$, where $\mathcal{L}'$ is the portion of $\mathcal{L}$ lying over N', we have $U' \subset U^{\operatorname{Exp}}$ and the normal exponential mapping sends U' diffeomorphically onto an open set in M.

The following classical result is also immediate from (c) in §14:

Gauss's Lemma. *Under these assumptions, all half-open geodesic segments of length ℓ, emanating from N' in directions normal to N, intersect orthogonally the* Exp*-images of all level sets of the norm function restricted to U'.* ∎

We can now prove the main result of this section.

Lemma 9.1. *For a special Kähler-Ricci potential τ on a Kähler manifold (M, g), let $\psi : M \to \mathbf{R}$ be defined as in* Lemma 7.2, *and let $Q = |\nabla\tau|^2$.*

Every point of M then has a neighborhood U on which Q is a C^∞ function of τ, that is, a composite consisting of τ followed by a C^∞ function $\tau \mapsto Q$ defined on a suitable interval of the variable τ, and such that $dQ/d\tau = 2\psi$ for $dQ/d\tau$ and ψ treated as functions on U.

Proof. At points with $d\tau \neq 0$ our assertion is obvious as $dQ = 2\psi\, d\tau$ (see (3.2.b)). Suppose now that $y \in M$ is a critical point of τ, and let N be the critical manifold of τ containing y (cf. Remark 2.3(iii)). We may choose N', ℓ, U' as in the third paragraph of this section and, making N' and ℓ smaller if necessary, also require that $d\tau \neq 0$ at every point of $\mathrm{Exp}(U' \smallsetminus N')$. (See (a) in Remark 2.3(iii).)

The gradients $v = \nabla\tau$ and $\nabla Q = 2\psi v$ (see (3.2.b)), which, by Lemma 8.3(i), are tangent to the geodesic segments mentioned in Gauss's Lemma, must therefore be normal to the Exp-images of all level sets of the norm function restricted to U'. Any such level set is either the zero section N', that is, the zero-level set, or it is a bundle of positive-dimensional spheres over N' (cf. the inequality in (c) of Remark 2.3(iii) for $u = J(\nabla\tau)$); therefore, it is connected, and so τ, Q must both be constant along its Exp-image. Thus, both τ and Q, restricted to $\mathrm{Exp}(U')$ and then pulled back to U' via Exp, are functions of the norm function.

Let $(-\ell, \ell) \ni s \mapsto x(s) \in M$ be any unit-speed geodesic such that $x(0) \in N'$ and $\dot{x}(0)$ is normal to N at $x(0)$, where $\dot{x} = dx/ds$. As $x(s) = \mathrm{Exp}(x(0), s\dot{x}(0))$ and the value of the norm function at $(x(0), s\dot{x}(0))$ is $|s|$, it follows that τ, Q treated as C^∞ functions of the variable $s \in (-\ell, \ell)$ (via the substitutions $\tau(x(s)), Q(x(s))$) are even. Their restrictions to $[0, \ell)$ express the dependence of their Exp-pull-backs on the norm function (also denoted by s). By Lemma 8.3(ii), $s \mapsto \tau$ is a homeomorphism, that is, Q restricted to $\mathrm{Exp}(U')$ is also a function of τ. Finally, $d^2\tau/ds^2 \neq 0$ at $s = 0$ in view of Remark 7.1(ii), since, by (7.1) and (2.1.iii), $\dot{x}(0)$ is an eigenvector of $(\nabla d\tau)(y)$ for the eigenvalue $a \neq 0$. That our assertion now follows also for the point y is clear since, if τ and Q are C^∞-differentiable even functions on an interval of a real variable s, centered at 0, and $d^2\tau/ds^2 \neq 0$ at $s = 0$, then Q, on some neighborhood of 0, is a C^∞ function of τ.

In fact, by induction on $k \geq 0$, any even C^{2k} function of s is a C^k function of $\zeta = s^2$. (Namely, if our claim holds for k, an even C^{2k+2} function f of s is necessarily of class C^{k+1} in ζ, as $\dot{f}(s)/s$, with $\dot{f} = df/ds$, is an even C^{2k} function of s, also at $s = 0$, due to Remark 4.2; hence, by the inductive assumption, $2\, df/d\zeta = \dot{f}(s)/s$ is a C^k function of ζ.) Thus, τ and Q are C^∞ functions of $\zeta = s^2$, while $\zeta \mapsto \tau$ is C^∞-diffeomorphic for $\zeta \geq 0$ close to 0, since $2d\tau/d\zeta = \dot{\tau}/s \to \ddot{\tau}(0) \neq 0$ as $s \to 0$ (that is, $\zeta \to 0$). This completes the proof. ■

§10. Isometric actions of the circle

The results of this section are two corollaries about special Kähler-Ricci potentials τ on compact Kähler manifolds. One, proved under more general assumptions, states that the Killing field $u = J(\nabla\tau)$ generates an isometric action of the circle; the other amounts to what will later become a boundary condition satisfied by $dQ/d\tau = 2\psi$, where $Q = |\nabla\tau|^2$ is treated as a function of τ (cf. Lemma 9.1).

For a C^∞ function τ on a Riemannian manifold (M, g), let $\mathrm{Crit}^1(\tau)$ be the set of those critical points y of τ at which the Hessian $\mathrm{Hess}_y\tau$ has exactly one nonzero eigenvalue (of any multiplicity). Thus,

$$\mathrm{Hess}_y\tau \text{ is semidefinite for every } y \in \mathrm{Crit}^1(\tau). \tag{10.1}$$

Lemma 10.1. *If a Killing field u on a Riemannian manifold (M, g) vanishes at a point y and $z \in T_yM$ lies in the domain of $\exp_y$, then $u(x)$, for $x = \exp_y z$, is the image of $\nabla_z u \in T_yM = T_z(T_yM)$ under the differential of $\exp_y$ at z.*

In fact, the local isometries Φ^t forming the local flow of u are all defined, for t near 0 in $\mathbf{R}$, on some open set in M containing the compact geodesic segment $X = \{\exp_y sz : 0 \le s \le 1\}$. Since they keep y fixed and map geodesics onto geodesics, we have $\Phi^t(\exp_y z) = \exp_y(d\Phi_y^t z)$. Our claim follows if we apply d/dt and let $t \to 0$, since $(\nabla u)(y) : T_yM \to T_yM$ is the infinitesimal generator of the one-parameter group $t \mapsto d\Phi_y^t$ in T_yM. ■

Lemma 10.2. *Suppose that τ is a Killing potential on a Kähler manifold (M, g) and a point $y \in M$ lies in the set $\mathrm{Crit}^1(\tau)$ defined above, while $v = \nabla\tau$ and $u = Jv$. In addition, let $U \subset M$ and $U' \subset T_yM$ be open metric balls centered at y and 0, respectively, and such that the exponential mapping $\exp_y$ sends U' diffeomorphically onto U, while the flow of u restricted to U consists of "global" isometries $U \to U$.*

Then the flow of u in U is periodic, with the minimum period $2\pi/|a|$ for the nonzero eigenvalue a of $\mathrm{Hess}_y\tau$.

This is immediate since, by Lemma 10.1, u restricted to U is the $\exp_y$-image of the linear vector field on U' given by the skew-adjoint (and hence diagonalizable) complex-linear operator $T_yM \ni z \mapsto \nabla_z u \in T_yM$ with the eigenvalues ai and 0, or just ai. (Namely, as $u = Jv$ and $\nabla J = 0$, its eigenvalues are i times those of $z \mapsto \nabla_z v$, that is, i times those of $(\nabla d\tau)(y)$, cf. (2.1.iii).) ■

Corollary 10.3. *Let $\tau : M \to \mathbf{R}$ be a Killing potential on a complete Kähler manifold (M, g) such that the set $\mathrm{Crit}^1(\tau)$ defined above is nonempty. Then*

- (i) *the flow of the Killing vector field $u = J(\nabla\tau)$ is periodic, that is, constitutes an isometric S^1 action on (M, g),*
- (ii) *the absolute value of the nonzero eigenvalue of $\mathrm{Hess}_y\tau$ is the same at all points $y \in \mathrm{Crit}^1(\tau)$.*

In fact, Lemma 10.2 and the unique continuation property for isometries ([**12**], Lemma 4 in §6 of Ch. VI) give (i); now $|a|$ in Lemma 10.2 is the same for all y. (The existence of U, U' in Lemma 10.2 is well known; cf. [**7**], Lemma 12.1(i).) ■

Corollary 10.4. *For a special Kähler-Ricci potential τ on a complete Kähler manifold (M, g), let $\psi : M \to \mathbf{R}$ be defined as in* Lemma 7.2. *Then the restriction of $|\psi|$ to the set $\mathrm{Crit}(\tau)$ of critical points of τ is constant and positive.* ■

This follows from Corollary 10.3(ii), since, by Lemma 7.2, the nonzero eigenvalue of $\mathrm{Hess}_y\tau$ at any $y \in \mathrm{Crit}(\tau) = \mathrm{Crit}^1(\tau)$ is $\psi(y)$. (Constancy of ψ on connected components of $\mathrm{Crit}(\tau)$ is also clear from Remark 7.4.) ■

§11. Boundary conditions in the compact case

We will now reach a second major step needed for a classification, in §16, of special Kähler-Ricci potentials τ on compact Kähler manifolds. The result in question is Proposition 11.5, and it states that τ has precisely two critical manifolds and $Q = |\nabla\tau|^2$ is, globally, a C^∞ function of τ, satisfying the positivity-and-boundary conditions (5.1). To prove it, we use some facts about Morse-Bott functions, of which Killing potentials on Kähler manifolds are a special case.

A C^∞ function $\tau : M \to \mathbf{R}$ on a manifold M is called a *Morse-Bott function* [**5**] if every connected component N of the set $\mathrm{Crit}(\tau)$ of its critical points satisfies conditions (a), (b) of Remark 2.3(iii) and, for every $y \in N$, the nullspace of the Hessian $\mathrm{Hess}_y\tau$ coincides with T_yN. (Since the nullspace contains T_yN for any submanifold N of M contained in $\mathrm{Crit}(\tau)$, the last requirement amounts to $\mathrm{rank}\,\mathrm{Hess}_y\tau = \dim M - \dim N$.) As in §2, the connected components of $\mathrm{Crit}(\tau)$ then are called the *critical manifolds* of τ.

Example 11.1. Every Killing potential τ on a Kähler manifold (Remark 2.1) is a Morse-Bott function: if τ is nonconstant, this follows from Remark 2.3(iii) and (2.1.iii), since $\mathrm{Hess}_y\tau = (\nabla d\tau)(y)$ whenever $y \in \mathrm{Crit}(\tau)$.

Lemma 11.2. *Let τ be a Morse-Bott function on a manifold M. Every point y of any critical manifold N of τ at which $\mathrm{Hess}_y\tau$ is positive or, respectively, negative semidefinite then has a neighborhood U such that $\tau > \tau(y)$ everywhere in $U \smallsetminus N$ or, respectively, $\tau < \tau(y)$ everywhere in $U \smallsetminus N$.*

This is clear as the Morse lemma ([**15**, p. 6]) has an obvious extension to Morse-Bott functions: in suitable local coordinates, y and τ appear as $(0,\dots,0)$ and, respectively, a homogeneous quadratic function plus a constant. ■

Lemma 11.3. *Let τ be a C^∞ function on a manifold M' such that the τ-preimage of every real number is compact and τ has no critical points. Then*

(i) *there exist a compact manifold P and a diffeomorphic identification $M' = P \times (\tau_-, \tau_+)$ under which τ appears as the projection onto the (τ_-, τ_+) factor, τ_- and τ_+ being the infimum and supremum of τ,*

(ii) *the τ-preimage of every real number is both compact and connected.*

In fact, the surjective submersion $\tau : M' \to (\tau_-, \tau_+)$, having compact fibres, is necessarily a locally trivial fibration, and hence a trivial bundle, as its base is contractible. (The inferences just used are both well known, and easily obtained, in our case, with the aid of the holonomy of any C^∞ connection, that is, a distribution complementary to the fibres.) This yields (i), and then (ii) follows. ■

Proposition 11.4. *Let τ be a Morse-Bott function on a compact manifold M such that $\mathrm{Hess}_y\tau$ is semidefinite for every $y \in \mathrm{Crit}(\tau)$, and the real codimensions of all critical manifolds of τ are greater than one. Then τ has exactly two critical manifolds, which are the τ-preimages of its extremum values $\tau_+ = \tau_{\max}$ and $\tau_- = \tau_{\min}$, and the τ-preimage of every real number is both compact and connected.*

Proof. As M is compact, τ has finitely many critical manifolds due to their being mutually isolated (cf. (a) in Remark 2.3(iii)), and none of them disconnects M, even locally (by the codimension condition), while τ is constant on each of them. Therefore, $M' = M \smallsetminus \mathrm{Crit}(\tau)$ is connected and dense in M, and the τ-image $\tau(M')$ is connected, open in $\mathbf{R}$, and dense in $[\tau_-, \tau_+]$, so that $\tau(M') = (\tau_-, \tau_+)$.

Moreover, the function $\tau : M' \to \mathbf{R}$ satisfies the hypotheses of Lemma 11.3. Namely, any sequence of points in M' that lies in the τ-preimage of a given real number has a subsequence converging to a limit $y \in M$, and then $y \notin \mathrm{Crit}(\tau)$, for otherwise our semidefiniteness assumption, combined with Lemma 11.2, would lead to a contradiction. Hence assertion (ii) in Lemma 11.3 holds for $\tau : M' \to \mathbf{R}$.

The only critical values of $\tau : M \to \mathbf{R}$ are $\tau_\pm$. In fact, let $y \in \mathrm{Crit}(\tau)$. Denseness of M' in M gives $x_k \to y$ as $k \to \infty$ for some sequence x_k in M'. If we had $\tau(y) \in (\tau_-, \tau_+)$, the sequence in $P \times (\tau_-, \tau_+)$ corresponding to the x_k under the identification of Lemma 11.3(i) (applied to $\tau : M' \to \mathbf{R}$, with $\tau(M') = (\tau_-, \tau_+)$) would have a convergent subsequence, that is, a subsequence of the x_k would have a limit in M', even though $x_k \to y \notin M'$.

Finally, connectedness of the τ-preimages $P[\tau']$ of real numbers τ', already established for $\tau' \neq \tau_\pm$, holds for $P[\tau_\pm]$ as well. In fact, given $\tau_0 = \tau_\pm$, let $N_1, \ldots, N_l$ be the connected components of $P[\tau_0]$. Also, let $U_1, \ldots, U_l$ be pairwise disjoint open sets in M with $N_j = U_j \cap \mathrm{Crit}(\tau)$ for $j = 1, \ldots, l$. The τ-preimage $P[\tau']$ of every $\tau' \in (\tau_-, \tau_+)$ sufficiently close to τ_0 must now be contained in the union $U = U_1 \cup \cdots \cup U_l$, or else there would be a sequence x_k in $M' \smallsetminus U$ with $\tau(x_k) \to \tau_0$ as $k \to \infty$, a subsequence of which would have a limit that lies in $P[\tau_0]$, yet not in the open set U containing $P[\tau_0]$. However, $P[\tau']$ obviously intersects *each* of the sets $U_1, \ldots, U_l$, for any $\tau' \in (\tau_-, \tau_+)$ sufficiently close to τ_0. Since such $P[\tau']$ are connected (see above) and $U_1, \ldots, U_l$ are pairwise disjoint and open, we must have $l = 1$. This completes the proof. ■

It is the assertion (i) in the next proposition that allows us to divide all triples (M, g, τ) with the stated properties into Class 1, characterized by case (1) in (i), and Class 2, for which (2) holds.

Proposition 11.5. *Let τ be a special Kähler-Ricci potential on a compact Kähler manifold (M, g) of complex dimension $m \geq 1$. Then*

- (i) *τ has exactly two critical manifolds, which are the τ-preimages of its extremum values $\tau_{\max}$ and $\tau_{\min}$, and one of two cases must occur*:
  - (1) *both critical manifolds of τ are of complex codimension one*;
  - (2) *one critical manifold of τ is of complex codimension one, and the other consists of a single point.*
- (ii) *In addition, $Q = |\nabla\tau|^2$ is a composite consisting of τ followed by a C^∞ function $[\tau_{\min}, \tau_{\max}] \ni \tau \mapsto Q \in \mathbf{R}$ that satisfies the positivity-and-boundary conditions* (5.1).

Proof. Our M and τ satisfy the assumptions of Proposition 11.4. (This follows from Example 11.1, the inequality in (c) of Remark 2.3(iii), and (a) in (10.1) combined with relation $\mathrm{Crit}(\tau) = \mathrm{Crit}^1(\tau)$, obvious from Proposition 7.3.) Now (i) is immediate from Propositions 11.4 and 7.3. In fact, unless $m = 1$, the critical manifolds of τ cannot both consist of single points, for if they did, Lemma 7.5 would give $\tau_{\max} = \tau_{\min} = c$, contrary to the requirement, in (1.1), that τ be non-constant. Another reason is that, if both critical manifolds were single points and we had $m \geq 2$, Reeb's theorem ([**15**, p. 25]) would imply that M is a topological n-sphere, $n \geq 4$, admitting no Kähler metric.

In view of (i), the open set $M' \subset M$ on which $d\tau \neq 0$ is the union of the τ-preimages of all values in $(\tau_{\min}, \tau_{\max})$. By Lemma 9.1, $Q = |\nabla\tau|^2$ is locally constant on every such τ-preimage; the word 'locally' may now be dropped as the τ-preimage is connected in view of Proposition 11.4, the assumptions of which hold, as we saw, in our case. Now (ii) follows, except for conditions (5.1), since C^∞-differentiability of the function $[\tau_{\min}, \tau_{\max}] \to \mathbf{R}$ is obvious from Lemma 9.1.

Lemma 9.1 also gives $2\psi = dQ/d\tau$ on the interval $[\tau_{\min}, \tau_{\max}]$. By Corollary 10.4, $|dQ/d\tau|$ now has the same positive value at both endpoints $\tau_{\min}, \tau_{\max}$. Finally, $dQ/d\tau > 0$ at $\tau_{\min}$ and $dQ/d\tau < 0$ at $\tau_{\max}$, as the function $\tau \mapsto Q = |\nabla\tau|^2$ is positive on the set $(\tau_{\min}, \tau_{\max})$ of non-critical values of τ (cf. (i)), and vanishes at $\tau_{\min}$ and $\tau_{\max}$. This yields (ii), completing the proof. ■

§12. Critical manifolds and curvature

The main result of this section, Lemma 12.4, establishes some curvature properties of critical manifolds of special Kähler-Ricci potentials, needed for our classification argument in §16.

Lemma 12.1. *Given a special Kähler-Ricci potential τ on a Kähler manifold (M, g), let ϕ, ψ be the C^∞ functions, introduced in §3, on the open set M' on which $d\tau \neq 0$. Then, for any C^∞ vector fields w, w' defined on an open subset of M' and orthogonal to $v = \nabla\tau$ and $u = Jv$ at every point, with R denoting the curvature tensor, $Q = |\nabla\tau|^2$, and $...^{\mathrm{vrt}}$ standing for the $\mathcal{V}$ component relative to the decomposition $TM' = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \mathrm{Span}\,\{v, u\}$ and $\mathcal{H} = \mathcal{V}^\perp$,*

- (i) $QR(w, w')v = 2(\phi - \psi)\phi g(Jw, w')u$,
- (ii) *either of $g(R(w, v)w', v)$ and $g(R(w, u)w', u)$ equals $g(w, w')$ times a function which does not depend on the choice of w and w',*
- (iii) $Q[\nabla_w w']^{\mathrm{vrt}} = -\phi[g(w, w')v + g(Jw, w')u]$,
- (iv) $Q[w, w']^{\mathrm{vrt}} = -2\phi\, g(Jw, w')u$.

Proof. Assertion (iii) was proved in [**7**], formula (13.1), and (iv) is obvious from (iii). Next, by (3.2.b), ϕ is constant in the direction of w, and hence (3.2.a) gives both $\nabla_w\nabla_{w'}v = \phi\nabla_w w'$ and $\nabla_{[w,w']}v = (\psi - \phi)[w, w']^{\mathrm{vrt}} + \phi[w, w']$. Now (i) easily follows from (2.1.i) and (iv). Next, writing $\langle\,,\rangle$ for $g(\ ,\ )$ we have

$$\langle\nabla_v\nabla_w w', v\rangle = d_v\langle\nabla_w w', v\rangle - \langle\nabla_w w', \nabla_v v\rangle = (\phi\psi - d_v\phi)\langle w, w'\rangle - \phi d_v\langle w, w'\rangle,$$

since $\nabla_v v = \psi v$ (by (3.2.a) for $w = v$) and $\langle\nabla_w w', v\rangle = \langle[\nabla_w w']^{\mathrm{vrt}}, v\rangle = -\phi\langle w, w'\rangle$ (from (iii) with $Q = |v|^2$), while $\langle\nabla_v w', v\rangle = -\langle w', \nabla_v v\rangle = 0$ (as $\nabla_v v = \psi v$), and so $\langle\nabla_w\nabla_v w', v\rangle = -\langle\nabla_v w', \nabla_w v\rangle = -\phi\langle w, \nabla_v w'\rangle$ (since (3.2.a) gives $\nabla_w v = \phi w$). As the local flow of v leaves $\mathcal{H}$ invariant (see [**7**], Remark 17.3, discussion of condition (a)), $[w, v]$ is a section of $\mathcal{H}$ and (3.2.a) gives $\nabla_w v = \phi w$, $\nabla_{[w,v]}v = \phi[w, v]$. Hence, as ∇ is torsion-free, $\langle\nabla_{[w,v]}w', v\rangle = -\langle w', \nabla_{[w,v]}v\rangle = \phi\langle w', [v, w]\rangle = \phi\langle w', \nabla_v w\rangle - \phi^2\langle w, w'\rangle$. Now (2.1.i) with $d_v\langle w, w'\rangle = \langle\nabla_v w, w'\rangle + \langle w, \nabla_v w'\rangle$ yields (ii) for $\langle R(w, v)w', v\rangle$. However, $\langle R(w, u)w', u\rangle = \langle R(Jw, v)Jw', v\rangle$ (and so (ii) for $\langle R(w, u)w', u\rangle$ follows as $\langle Jw, Jw'\rangle = \langle w, w'\rangle$). Namely, ∇ is a connection in the *complex* vector bundle TM, since $\nabla J = 0$, and so the operator $w' \mapsto R(w, v)w'$ commutes with J. Thus, as J is skew-adjoint, $\langle R(w, u)w', u\rangle = \langle R(w, u)w', Jv\rangle = -\langle R(w, u)Jw', v\rangle$, which in turn equals $-\langle R(Jw', v)w, u\rangle = -\langle R(Jw', v)w, Jv\rangle = \langle R(Jw', v)Jw, v\rangle = \langle R(Jw, v)Jw', v\rangle$. This completes the proof. ■

Remark 12.2. The normal bundle $\mathcal{L}$ of any submanifold N of a Riemannian manifold (M, g) carries the usual *normal connection* ∇^{nrm}, characterized by $\nabla^{\mathrm{nrm}}_{\dot{y}}w = [\nabla_{\dot{y}}w]^{\mathrm{nrm}}$ whenever $t \mapsto w(t) \in T_{y(t)}M$ is a C^1 vector field normal to N along a C^1 curve $t \mapsto y(t) \in N$. Here $\dot{y} = dy/dt$ and ∇ is the Levi-Civita connection of (M, g), while $...^{\mathrm{nrm}}$ stands for the component normal to N.

Remark 12.3. Given a totally geodesic submanifold N of a Riemannian manifold (M,g), a point $y \in N$, and vectors $w, w' \in T_yN$, let ∇, R be the Levi-Civita connection and curvature tensor of (M,g), and let a Riemannian metric h on N be a constant multiple of its submanifold metric. Then, for any vector ξ tangent (or, normal) to N at y, the value $R(w,w')\xi$ coincides with the one obtained by replacing R with the curvature tensor of (N,h) (or, respectively, the curvature tensor of the normal connection ∇^{nrm} in the normal bundle $\mathcal{L}$ of N).

In fact, extending w, w', ξ to C^∞ vector fields on a neighborhood U of y in M tangent/normal to N along $N \cap U$, we see that $\nabla_w \xi$, restricted to $N \cap U$, is the covariant derivative relative to the Levi-Civita connection of (N,h) (or, respectively, the normal connection in $\mathcal{L}$), and our claim is obvious from (2.1.i).

Suppose that τ is a special Kähler-Ricci potential on a Kähler manifold (M,g) and N is a critical manifold of τ with $\dim_{\mathbf{C}} N = \dim_{\mathbf{C}} M - 1$ (cf. Proposition 7.3), while c, ε and a are the constants introduced in Lemma 3.1 and Proposition 7.3. If $\dim_{\mathbf{C}} M \ge 2$, we define a Kähler metric h on the complex manifold N to be g/f_0 restricted to TN, where f_0 is the constant given by $f_0 = 1$ (if $\varepsilon = 0$), or $f_0 = 2|\tau_0 - c|$ (if $\varepsilon = \pm 1$), with τ_0 denoting the constant value of τ on N. Thus, $f_0 > 0$, as $\tau \ne c$ on N when $\varepsilon = \pm 1$, in view of Lemma 7.5.

Lemma 12.4. *Let N be a critical manifold of a special Kähler-Ricci potential τ on a Kähler manifold (M,g) with $\dim_{\mathbf{C}} M = m \ge 1$ and $\dim_{\mathbf{C}} N = m-1$, and let $\mathcal{L}$ be the normal bundle of N. Then, for $\mathcal{H}, M'$ as in §3 and ε, a, h as above,*

- (a) *the limit relation $\mathcal{H}_x \to T_yN$ as $x \to y$, for any fixed $y \in N$ and a variable point $x \in M'$, holds in an appropriate Grassmannian bundle,*
- (b) *the Kähler manifold (N,h) is Einstein unless $m = 2$,*
- (c) *the curvature form Ω of the normal connection in $\mathcal{L}$ and the Kähler form $\omega^{(h)}$ of (N,h) are related by $\Omega = -2\varepsilon a \omega^{(h)}$.*

Proof. Since (a) – (c) hold trivially when $m = 1$, we may assume that $m \ge 2$.

First, (a) follows from Remark 7.4 and Lemma 7.2: as $x \to y$, the eigenvalue $\phi(x)$ of $(\nabla d\tau)(x)$ tends to the eigenvalue $\phi(y) = 0$ of $(\nabla d\tau)(x)$ having the same multiplicity $2(m-1)$, and so we have the convergence $\mathcal{H}_x \to T_yN$ of the corresponding eigenspaces. (Cf. (3.1), (d) in Remark 2.3(iii) and (2.1.iii).)

Let $f : M' \to \mathbf{R}$ be given by $f = 1$ if $\varepsilon = 0$ and $f = 2|\tau - c|$ if $\varepsilon = \pm 1$, so that $f > 0$ on M' by Lemma 3.1. Also, let $v = \nabla\tau$, $u = Jv$ and $Q = |\nabla\tau|^2$. For any $x \in M'$ we now define symmetric bilinear forms $h(x)$ and $\mathrm{r}^{(h)}(x)$ on the space $\mathcal{H}_x$ by declaring $h(x)$ to be the restriction of $g(x)/f(x)$ to $\mathcal{H}_x$, and letting $\mathrm{r}^{(h)}(x)$ assign to vectors $w, w' \in \mathcal{H}_x$ the value $\mathrm{r}^{(h)}(w,w')$ equal to $\sum_j g(R(w,e_j)w', e_j)$, where R is the curvature tensor of (M,g) and the e_j run through any $g(x)$-orthonormal basis of $\mathcal{H}_x$. Since such e_j along with $Q^{-1/2}v$ and $Q^{-1/2}u$ (at x) clearly form a $g(x)$-orthonormal basis of T_xM, our $\mathrm{r}^{(h)}(x)$ and the Ricci tensor $\mathrm{r}(x)$ of g at x are related by $\mathrm{r}^{(h)}(w,w') = \mathrm{r}(w,w') - [g(R(w,v)w',v) + g(R(w,u)w',u)]/Q$, for $w, w' \in \mathcal{H}_x$, with v, u, Q standing for their values at x. Since $\mathrm{r} = \lambda g$ on $\mathcal{H}$ (see (3.1)), Lemma 12.1(ii) shows that $\mathrm{r}^{(h)}(x)$ equals a scalar times $h(x)$.

As M' is dense in M (Remark 2.3(ii)), choosing a sequence of points $x \in M'$ converging to any given $y \in N$ and using (a) along with Remark 12.3, we see that the Ricci tensor of h at y is a scalar multiple of $h(y)$, which proves (b).

Equality $R(w,w')\xi = i\,\Omega(w,w')\xi$ for $w,w' \in \mathcal{H}_x$, $\xi \in \mathcal{H}_x^\perp$ and $x \in M'$, with $\Omega(w,w')$ equal to $2(\phi-\psi)\phi/Q$ at x times $g(Jw,w')$, is immediate: if $\xi = v(x)$, it follows from Lemma 12.1(i), and so it holds for all ξ as the operator $R(w,w') : T_xM \to T_xM$ commutes wih $J(x)$. Let a variable point $x \in M'$ now tend to any fixed $y \in N$. Then (a) implies the above equality also when x and $\mathcal{H}_x$ are replaced by y and T_yN, while $\phi(x) \to 0$, $\psi(x) \to a$ (see Remark 7.4) and, unless $\phi = 0$ identically, Lemma 3.1 yields $Q(x)/\phi(x) \to 2[\tau(y)-c] = \varepsilon f(y)$. Now (c) follows from Remark 12.3, completing the proof. ■

§13. A normal exponential diffeomorphism

This section contains the third major step needed for our classification of special Kähler-Ricci potentials τ on compact Kähler manifolds in §16. Namely, Lemma 13.2 describes a diffeomorphism that will later give rise to a composite diffeomorphism $\Psi = \mathrm{Exp}\circ\Phi : \mathcal{L}\smallsetminus N \to M'$ used in the proof of Theorem 16.3. Here $\mathcal{L}$ is the normal bundle of any given critical manifold N of τ, and $M' \subset M$ is the open set on which $d\tau \neq 0$.

First, we set $L = \int_{\tau_{\min}}^{\tau_{\max}} Q^{-1/2}\,d\tau$ for any given function $\tau \mapsto Q$ satisfying the positivity-and-boundary conditions (5.1) on an interval $[\tau_{\min},\tau_{\max}]$. Thus, $L < \infty$, as one sees making Q the variable of integration near either endpoint. Let τ now be a special Kähler-Ricci potential on a compact Kähler manifold. The next lemma shows that L, corresponding in this manner to the function $\tau \mapsto Q$ obtained in Proposition 11.5(ii), then is the minimum distance at which a normal geodesic emanating from a critical manifold of τ encounters another critical point of τ.

Lemma 13.1. *Let N, N^* be the two critical manifolds of a special Kähler-Ricci potential τ on a compact Kähler manifold (M,g), cf.* Proposition 11.5(ii). *Then, with $L = \int_{\tau_{\min}}^{\tau_{\max}} Q^{-1/2}\,d\tau$ for $Q = |\nabla\tau|^2$ treated as a function of τ,*

(a) *L is the minimum distance between N and any given point $y' \in N^*$,*
(b) *every point $x \in M$ at which $d\tau \neq 0$ can be joined to N by a geodesic, normal to N, of some length $\ell \in (0,L)$,*
(c) *for any geodesic $X \subset M$ of length L with endpoints y,y' such that $y \in N$ and X is normal to N at y, we have $y' \in N^*$ and $Q > 0$ on $X\smallsetminus\{y,y'\}$.*

Proof. For X, y, y' as in (c), let X' be the maximal half-open geodesic segment containing y, as an endpoint, along with all points of X sufficiently close to y, and such that $d\tau \neq 0$ everywhere in $X'\smallsetminus\{y\}$. Let $[0,\ell) \ni s \mapsto x(s)$ be an arc-length parameterization of X'. By Lemma 8.3(ii), $ds = \pm Q^{-1/2}\,d\tau$ on X', so that $\ell = \int_0^\ell ds = \left|\int_{\tau_0}^{\tau'} Q^{-1/2}\,d\tau\right|$, where $\tau_0 = \tau(y) \in \{\tau_{\min},\tau_{\max}\}$ is the value of τ on N, and $\tau' = \tau(x(\ell))$ with $x(\ell) = \lim_{s\to\ell} x(s)$. (Clearly, $\ell \le L < \infty$, and $\ell < L$ unless $\tau' \in \{\tau_{\min},\tau_{\max}\}$.) However, maximality of X' now shows that $(d\tau)(x(\ell)) = 0$, and so, as $\tau(x(\ell)) = \tau'$, Proposition 11.5(i) gives $\{\tau_0,\tau'\} = \{\tau_{\min},\tau_{\max}\}$, that is, $\ell = L$. Consequently, $X' = X$ and (c) follows.

Given $y' \in N^*$, let y be the point of N nearest to y', and let X' be a minimizing geodesic segment of some length L', joining y' to y. As (c) implies that every point in a given critical manifold lies at the distance L from some point in the other critical manifold, we have $L' \le L$. On the other hand, $L' \ge L$. In fact, if we had $L' < L$, by extending X' beyond y' so as to obtain a geodesic segment X of length L we would conclude, from the final clause of (c), that y' is not a critical point of τ. (Note that X' is normal to N at y due to our distance-minimizing choice of y and X'.) Hence $L' = L$, which gives (a).

To prove (b), let us connect any $x \in M' = M \smallsetminus (N \cup N^*)$ with the point y nearest to it in $N \cup N^*$ by a minimizing geodesic segment X' of some length $\ell > 0$. Thus, $\ell < L$, or else some point of X' would lie at the distance L from y, and so, by (c), it would be a point of $N \cup N^*$, closer to x than y is. Extending X' beyond x, we obtain a geodesic segment X of length L and, by (c), one of the endpoints of X lies in N. Also, X is normal to N at that endpoint, since, by (a), X is a minimum-length curve joining N to N^*. This completes the proof. ∎

Lemma 13.2. *Let* $L = \int_{\tau_{\min}}^{\tau_{\max}} Q^{-1/2}\, d\tau$ *for* $\tau_{\min}, \tau_{\max}$ *and* $\tau \mapsto Q$ *determined as in* Proposition 11.5(ii) *by a special Kähler-Ricci potential* τ *on a compact Kähler manifold* (M, g)*. If* $\mathcal{L}$ *denotes the total space of the normal bundle of a critical manifold* N *of* τ*, while* $M' \subset M$ *and* $\mathcal{L}' \subset \mathcal{L}$ *are the open sets defined by* $d\tau \neq 0$ *and, respectively,* $0 < s < L$*, where* s *stands for the norm function of the fibre metric obtained by restricting* g *to* $\mathcal{L}$*, then the normal exponential mapping of* N*, defined as in* §9*, is a* C^∞ *diffeomorphism* $\mathrm{Exp} : \mathcal{L}' \to M'$.

Proof. The Exp-image of any open line segment of length L emanating from 0 in any fibre $\mathcal{L}'_y$ of the punctured-disk bundle $\mathcal{L}'$ has the form $X \smallsetminus \{y, y'\}$, where X and y, y' satisfy the premise, and hence also the conclusion, of Lemma 13.1(c); thus, $X \smallsetminus \{y, y'\} \subset M'$, and so Exp actually sends $\mathcal{L}'$ into M'.

Surjectivity of $\mathrm{Exp} : \mathcal{L}' \to M'$ is obvious from Lemma 13.1(b). To prove its injectivity, suppose that $(y, z) \in \mathcal{L}'$ and $x = \mathrm{Exp}(y, z) \in M'$. Since $0 < |z| < L$, we can express (y, z) in terms of x by travelling backwards along the unit-speed geodesic $t \mapsto x(t) = \mathrm{Exp}(y, tz/|z|)$, which has $x(0) = y$, $\dot x(0) = z/|z|$, $x(s) = x$ (where $\dot x = dx/dt$ and $s = |z| \in (0, L)$) and, by Lemma 8.3(i), $\dot x(s) = w(x)$ for the vector field $w = (\operatorname{sgn} a)v/|v|$ on M' (with v, a as in Lemma 8.3). In fact, the re-parameterized geodesic $t \mapsto y(t) = \mathrm{Exp}(x, -tw(x))$ clearly has $y(0) = x$, $\dot y(0) = -w(x)$, $y(s) = y$ and $\dot y(s) = -z/|z|$, so that $(y, z) = (y(s), -s\dot y(s))$. Moreover, s is uniquely determined by x and depends C^∞-differentiably on x (via $\tau(x)$), since the assignment $s \mapsto \tau$ defined by condition (b) in Lemma 8.3 is a C^∞ diffeomorphism $(0, L) \to (\tau_{\min}, \tau_{\max})$. The last formula for (y, z) thus shows that (y, z) is determined by x, i.e., $\mathrm{Exp} : \mathcal{L}' \to M'$ is injective, and its inverse $M' \to \mathcal{L}'$ is of class C^∞. This completes the proof. ∎

§14. Variations and partial covariant derivatives

In this section we derive some equalities, needed in §15, and involving variations of normal geodesics emanating from a critical manifold of a special Kähler-Ricci potential on a Kähler manifold.

Let $(s, t) \mapsto x(s, t) \in M$ be a fixed C^∞ *variation of curves* in a manifold M, that is, a C^∞ mapping with real variables s, t ranging independently over some intervals. By (s, t)-*dependent* functions φ or vector fields w we then mean assignments sending each (s, t) to $\varphi(s, t) \in \mathbf{R}$ or $w(s, t) \in T_{x(s,t)}M$. Differentiability of such objects is well defined, as they are sections of specific pullback bundles. For instance, the velocities of the curves $s \mapsto x(s, t)$ and $t \mapsto x(s, t)$, with t or s fixed, are (s, t)-dependent vector fields, here denoted by x_s and x_t, that have, in local coordinates, the components $x_s^j = \partial x^j/\partial s$ and $x_t^j = \partial x^j/\partial t$, where $x^j(s,t)$ are the components of $x(s,t)$. Ordinary vector fields u or functions f on M give rise to (s,t)-dependent ones that assign $u(x(s,t))$ or $f(x(s,t))$ to any (s,t).

We use the subscript notation φ_s, φ_t for the partial derivatives of (s,t)-dependent C^1 functions φ, including ordinary C^1 functions on M. If, in addition, there is a fixed connection ∇ in the tangent bundle TM, we may differentiate (s,t)-dependent C^1 vector fields w covariantly with respect to either parameter s or t (that is, along the curves mentioned above), obtaining (s,t)-dependent fields w_s, w_t equal to $\nabla_{\dot x} w$ for $\dot x = x_s$ (or, $\dot x = x_t$), with the local-coordinate expressions $w_s^j = \partial w^j/\partial s + \Gamma_{kl}^j x_s^k w^l$ and $w_t^j = \partial w^j/\partial t + \Gamma_{kl}^j x_t^k w^l$. Here Γ_{kl}^j are the component functions of ∇, evaluated at $x(s,t)$, and repeated indices are summed over.

Applied to x_s and x_t, this leads to the (s,t)-dependent fields $x_{ss} = (x_s)_s$, $x_{st} = (x_s)_t$, etc. Thus, $x_{ss} = 0$ identically if and only if all the curves $s \mapsto x(s,t)$ are uniform-parameter geodesics. If ∇ is torsion-free, then $\Gamma_{kl}^j = \Gamma_{lk}^j$, and so

$(*)$ $x_{st} = x_{ts}$.

Let us now assume that ∇ is the Levi-Civita connection of a fixed Riemannian metric g on M, while N is a submanifold of M and $t \mapsto \zeta(t)$ is a C^∞ unit vector field normal to N along some given C^∞ curve $t \mapsto y(t) \in N$, where t ranges over some interval. Let us set $x(s,t) = \mathrm{Exp}(y(t), s\zeta(t))$ for all s in some interval of the form $[0,\ell]$ with $\ell > 0$, where $\mathrm{Exp} : U^{\mathrm{Exp}} \to M$ is defined as in Remark 8.1. (Such ℓ exists, that is, $(y(t), s\zeta(t)) \in U^{\mathrm{Exp}}$ for all s,t, provided that one replaces the original interval of t with a suitable subinterval.) Then

(a) $|x_s| = 1$ and $x_{ss} = 0$ for all s,t,
(b) $x_{st} = \nabla_{\dot y}\zeta$ at $s = 0$ and any t,
(c) $\langle x_s, x_t\rangle = 0$ for all s,t, which is known as *Gauss's Lemma* (cf. §9),
(d) $x_s(0,t) = \zeta(t)$ is unit and normal to N, and $x_t(0,t) = \dot y(t)$ is tangent to N, with $\dot y = dy/dt$,

where $\langle\,,\rangle$ stands for $g(\ ,\ )$. In fact, the formula for $x(s,t)$ implies (a), (b) and (d). The Leibniz rule for (s,t)-dependent functions such as $\langle x_s, x_t\rangle$ yields $\langle x_s, x_t\rangle_s = \langle x_{ss}, x_t\rangle + \langle x_s, x_{ts}\rangle$, and, from $(*)$, $2\langle x_s, x_{ts}\rangle = 2\langle x_s, x_{st}\rangle = \langle x_s, x_s\rangle_t$. Hence (a) gives $\langle x_s, x_t\rangle_s = 0$, that is, $\langle x_s, x_t\rangle$ does not depend on s, and (c) follows since, by (d), $\langle x_s, x_t\rangle = 0$ when $s = 0$.

Still making all the assumptions listed in the paragraph following $(*)$, let us also suppose that (M,g) is a Kähler manifold with a special Kähler-Ricci potential τ and N is a critical manifold of τ, while $v = \nabla\tau$, $u = Jv$, $Q = |\nabla\tau|^2$, and ϕ, ψ are the C^∞ functions, defined in §3, on the open set M' given by $d\tau \neq 0$. If $x(s,t) \in M'$ for all $s > 0$ and all t, then, for all s,t with $s > 0$,

(e) $|v| = |u| = Q^{1/2}$,
(f) $Q_s = \pm 2\psi Q^{1/2}$ and $\phi_s = \pm 2(\psi - \phi)\phi Q^{-1/2}$,
(g) $v = \pm Q^{1/2} x_s$,
(h) $\langle u, x_t\rangle_s = \pm 2\langle u, x_t\rangle \psi Q^{-1/2} = 2\langle u, x_{st}\rangle$,

where $\pm$ is the sign of the constant a in Proposition 7.3. In fact, (e) is obvious, while Lemma 8.3(i) and (a) give $v = \pm|v|x_s$ with the required sign $\pm$, so that (g) follows from (e). Also, $f_s = \langle x_s, \nabla f\rangle$ for $f = Q$ and $f = \phi$, and so (3.2.b), (g) and (a) yield (f). Next, $\langle u, x_{ts}\rangle = -\langle u_t, x_s\rangle = \langle u_s, x_t\rangle$. Namely, the first relation follows from the Leibniz rule and $(*)$, as $\langle u, x_s\rangle = 0$ (by (g), since $\langle u, v\rangle = 0$), while the second is clear from skew-symmetry of ∇u, as $u_s = (\nabla u)x_s$. The Leibniz rule now yields $\langle u, x_t\rangle_s = \langle u_s, x_t\rangle + \langle u, x_{ts}\rangle = 2\langle u_s, x_t\rangle$. This implies both $\langle u, x_t\rangle_s = 2\langle u, x_{ts}\rangle$ and $\langle u, x_t\rangle_s = \pm 2\langle u, x_t\rangle \psi Q^{-1/2}$ (since (3.2.a) gives $\nabla_v u = \nabla_v(Jv) = J\nabla_v v = \psi u$, so that, by (g), $u_s = \pm\psi Q^{-1/2}u$). Thus, (h) follows.

§15. The differential of the normal exponential mapping

This section provides a finishing touch required for the classification argument in §16. Namely, in the proof of Theorem 16.3 we use the normal bundle $\mathcal{L}$ of a critical manifold N of a special Kähler-Ricci potential τ on a compact Kähler manifold (M, g) to construct a Class 1 or Class 2 triple as in §5 or §6, and then exhibit a biholomorphic isometry $\varPsi$ between that triple and (M, g, τ). Specifically, $\varPsi$ is the composite of the normal exponential mapping Exp of $\mathcal{L}$ preceded by a suitable fibre-preserving mapping $\mathcal{L} \to \mathcal{L}$. Here we establish some properties of Exp needed for showing, in §16, that $\varPsi$ is in fact holomorphic and isometric.

Let us consider (the total space of) the normal bundle $\mathcal{L}$ of a fixed critical manifold N of a special Kähler-Ricci potential τ on a Kähler manifold (M, g). By Proposition 7.3, two cases are possible:

(I) N is a complex submanifold of complex codimension 1 in M, and so its normal bundle $\mathcal{L}$ is a complex line bundle over N, or

(II) $N = \{y\}$ for some point $y \in M$, so that $\mathcal{L} = \{y\} \times T_yM$.

We denote by $\mathcal{H}^N$ the distribution on $\mathcal{L} \smallsetminus N$ such that, in case (I), $\mathcal{H}^N$ is the restriction to $\mathcal{L} \smallsetminus N$ of the horizontal distribution of the normal connection in $\mathcal{L}$ (cf. Remark 12.2), and, in case (II), $\mathcal{H}^N$ is the distribution on $T_yM \smallsetminus \{0\}$ with $\mathcal{H}^N_w = w^\perp \cap (Jw)^\perp$ for any $w \in T_yM \smallsetminus \{0\}$, where $\mathcal{L} = \{y\} \times T_yM$ is identified with T_yM. Note that $\mathcal{H}^N$ is not only a real vector subbundle of the tangent bundle $T(\mathcal{L} \smallsetminus N)$, but also a *complex* vector bundle, with the complex structure in each fibre $\mathcal{H}_* = \mathcal{H}^N_{(y,z)}$ inherited, in case (II), from the ambient space T_yN (in which $\mathcal{H}_*$ is contained as a complex subspace), or pulled back, in case (I), from T_yN by the differential at (y, z) of the bundle projection $\mathcal{L} \to N$.

We also define vector fields v^N, u^N on $\mathcal{L} \smallsetminus N$ by $v^N(y, z) = az$ and $u^N(y, z) = iaz$ for $y \in N$ and $z \in (T_yN)^\perp$, with a determined by N as in Proposition 7.3.

Suppose now that $\mathcal{L}$ is, again, the total space of the normal bundle of a critical manifold N of a special Kähler-Ricci potential τ on a Kähler manifold (M, g) with $\dim_{\mathbf{C}} M = m \geq 1$, while $\mathcal{H}^N, v^N$ and u^N are the complex vector bundle and vector fields defined above. Similarly, in the open set $M' \subset M$ on which $d\tau \neq 0$ we have the vector fields $v = \nabla\tau$, $u = Jv$ and the distribution $\mathcal{H} = v^\perp \cap u^\perp$. We also assume that the image of the set $U' = (U^{\mathrm{Exp}} \cap \mathcal{L}) \smallsetminus N$ under the normal exponential mapping Exp of N, cf. §9, is contained in M' (which can always be achieved by removing from M all critical manifolds of τ other than N).

Lemma 15.1. *Under these assumptions, let $\varTheta : T_{(y,z)}\mathcal{L} \to T_xM$ denote the differential of* Exp *at some fixed $(y, z) \in U'$, with $x = \mathrm{Exp}(y, z)$, and let the symbol $\mathcal{H}_*$ stand for $\mathcal{H}^N_{(y,z)}$. Then $\varTheta(\mathcal{H}_*) \subset \mathcal{H}_x$ and $\varTheta : \mathcal{H}_* \to \mathcal{H}_x$ is complex-linear. Also, letting $w, w' \in T_yN$ be the images of any given $\xi, \xi' \in \mathcal{H}_*$ under the differential at (y, z) of the bundle projection $\mathcal{L} \to N$, we have, with c, ε as in* Lemma 3.1,

(i) $g(\varTheta\xi, \varTheta\xi') = g(w, w')$ *if* $\varepsilon = 0$,

(ii) $[\tau(y) - c]\, g(\varTheta\xi, \varTheta\xi') = [\tau(x) - c]\, g(w, w')$ *in case* (I), *if* $\varepsilon \neq 0$,

(iii) $a g(z, z)\, g(\varTheta\xi, \varTheta\xi') = 2\,[\tau(x) - c]\, g(\xi, \xi')$ *in case* (II), *if* $\varepsilon \neq 0$, *where a is the constant defined, for N, in* Proposition 7.3. *Finally, the $\varTheta$-images of $v^{\mathrm{N}}(y,z)$ and $u^{\mathrm{N}}(y,z)$ are $|az|v(x)/|v(x)|$ and, respectively, $u(x)$.*

Proof. Let $y(t),\zeta(t),x(s,t)$ be as in the paragraph following $(*)$ in §14 and, in addition, such that, in case (I) above, the unit vector field $t\mapsto\zeta(t)$ normal to N along the curve $t\mapsto y(t)\in N$ is parallel relative to the Levi-Civita connection of (M,g), while, in case (II), $y(t)=y$ for all t and $\dot\zeta=d\zeta/dt\in T_yM$ is $g(y)$-orthogonal to $\zeta(t)$ and $J\zeta(t)$ for every t. These assumptions mean that, for any fixed s, the curve $t\mapsto(y(t),s\zeta(t))$ in U' is "horizontal" (tangent to $\mathcal{H}^N$ at every point).

Note that every vector in $\mathcal{H}^N$ at any point of $\mathcal{L}\smallsetminus N$ is tangent to a horizontal curve, since our $\mathcal{H}^N$ is, also in case (II), the horizontal distribution of a connection (cf. §6). Also, in case (I) we assume that $\nabla_{\dot y}\zeta=0$, rather than just $[\nabla_{\dot y}\zeta]^{\mathrm{nrm}}=0$ as required by the definition of the normal connection (Remark 12.2), since N is totally geodesic, cf. Remark 2.3(iii)(c), and so $\nabla_{\dot y}\zeta$ is normal to N whenever ζ is.

Writing $\langle\,,\rangle$ for $g(\,,\,)$ we have $\langle v,x_t\rangle=\langle u,x_t\rangle=0$ for all s,t (notation of §14). First, $\langle v,x_t\rangle=0$ by (g), (c) in §14. Next, (h) and (f) in §14 yield $[\langle u,x_t\rangle/Q]_s=0$, that is, $\langle u,x_t\rangle/Q$ is constant as a function of s with fixed t. To see that its constant value is 0, we evaluate its limit as $s\to0$ using l'Hospital's rule and noting that, by (h), (f) in §14, $\langle u,x_t\rangle_s/Q_s=\pm\langle u,x_{st}\rangle Q^{-1/2}/\psi$. In case (I), $\langle u,x_t\rangle_s/Q_s=\pm\langle u/|u|,x_{st}\rangle/\psi\to0$ as $s=0$, by (b), (e) in §14, since $\nabla_{\dot y}\zeta=0$, while $\psi=a\neq0$ on N (see Remark 7.4). In case (II), $\langle u,x_t\rangle_s/Q_s=\langle Jx_s,x_{st}\rangle/\psi\to\langle J\zeta(t),\dot\zeta\rangle/a=0$ as $s\to0$ by (g), (c) in §14 with $u=Jv$, $\psi=a\neq0$ on N, and our orthogonality assumption for case (II). Thus, $\langle u,x_t\rangle=0$. Hence $\varTheta(\mathcal{H}_*)\subset\mathcal{H}_x$, as $\langle v,x_t\rangle=\langle u,x_t\rangle=0$, that is, v and u are g-normal to the Exp-image of every horizontal curve in U'.

To prove (i) – (iii) we may assume, due to symmetry of g, that $\xi=\xi'$. Since, as we just saw, $x_t(s,t)\in\mathcal{H}_{x(s,t)}$, while v_t is the covariant derivative of v in the direction of x_t (cf. §14), (3.2.a) yields $v_t=\phi x_t$ for every (s,t). Also, $Q_t=\langle x_t,\nabla Q\rangle=0$, as $\nabla Q=2\psi v$ (see (3.2.b)) and $\langle v,x_t\rangle=0$. Thus, $x_{st}=\pm Q^{-1/2}\phi x_t$, since $x_s=\pm Q^{-1/2}v$ by (g) in §14. The Leibniz rule and $(*)$ in §14 now give $\langle x_t,x_t\rangle_s=2\langle x_t,x_{st}\rangle=\pm2\langle x_t,x_t\rangle\phi Q^{-1/2}$. This, along with (d) in §14, yields $\langle x_t,x_t\rangle=g(\dot y,\dot y)$ when $\phi=0$, thus proving (i); at the same time, combined with (f) in §14, it implies that $[\langle x_t,x_t\rangle\phi/Q]_s=0$, and so $\langle x_t,x_t\rangle\phi/Q$ does not depend on s. When $\phi\neq0$ on M', Lemma 3.1 gives $Q/\phi=2(\tau-c)$, so that $\langle x_t,x_t\rangle/(\tau-c)$ is constant as a function of s, and we find its value by taking its limit as $s\to0$. Specifically, in case (I), $\tau(y)\neq c$ (see Lemma 7.5(ii)), and (ii) follows as $\langle x_t,x_t\rangle/(\tau-c)$ at $x=x(s,t)$ equals $g(\dot y,\dot y)/(\tau-c)$ at $y=x(0,t)$, cf. (d) in §14. In case (II) we find the limit by using l'Hospital's rule *twice*, as $\tau(y)=c$ (Lemma 7.5(ii)) and $d\tau=0$ at y, while $x_t=0$ at $s=0$ by (d) in §14; this gives $2\langle x_{ts},x_{ts}\rangle$ in the numerator (at $s=0$) and τ_{ss} in the denominator. By Remark 7.1(ii), (7.1) and $(*)$, (b) in §14, $\tau_{ss}=\psi(y)=a$ and $x_{ts}=\dot\zeta$ at $s=0$. (In this case $\nabla_{\dot y}\zeta$ really stands for $\dot\zeta$, as $y(t)=y$ is constant.) Now (iii) follows: $\langle x_t,x_t\rangle/(\tau-c)$ at any $x=x(s,t)$ is the same as at $y=x(0,t)$, that is, $\langle x_t,x_t\rangle/(\tau-c)=2g(\dot\zeta,\dot\zeta)/a=2g(\xi,\xi)/(as^2)=2g(\xi,\xi)/[ag(z,z)]$ for $z=s\zeta(t)$ and $\xi=s\dot\zeta(t)$.

Equality $x_{st}=\pm Q^{-1/2}\phi x_t$ obtained above amounts to $\nabla_{\dot x}w=\pm Q^{-1/2}\phi w$ for $\dot x=x_s$, where $w=x_t$ stands for the vector field $s\mapsto w(s)=x_t(s,t)$ along the geodesic $s \mapsto x(s,t)$, with fixed t. As $\nabla J = 0$, relation $\nabla_{\dot{x}} w = \pm Q^{-1/2}\phi w$ holds for $\tilde{w} = Jw$ whenever it does for w. In case (I), w has an Exp-preimage which is a vector field along the curve $s \mapsto (y(t), s\zeta(t)) \in \mathcal{L}$ arising from the horizontal lift of $w(0)$. (In fact, at any s, t, the preimage is the velocity vector of the curve $t \mapsto (y(t), s\zeta(t))$, which we chose to be horizontal, and which has the projection image $t \mapsto y(t)$ with the velocity $w(0)$, cf. (d) in §14.) Replacing $w(0)$ by $Jw(0)$ causes such a horizontal-lift field to become multiplied by i in the complex vector bundle $\mathcal{H}^N$, and at the same time results in replacing w such that $\nabla_{\dot{x}} w = \pm Q^{-1/2}\phi w$ for $\dot{x} = x_s$ by $\tilde{w} = Jw$, since w then is determined by the initial value $w(0)$. Thus, $\Theta : \mathcal{H}_* \to \mathcal{H}_x$ is complex-linear in case (I). In case (II), with $\mathcal{L} = T_y M$, an Exp-preimage of w is the vector field $s \mapsto s\dot{\zeta}(t)$ along the line segment $s \mapsto s\zeta(t) \in T_y M$ (where t is fixed). Hence $w(0) = 0$ and $w(s)/s$ has a limit as $s \to 0$, equal, by the local-coordinate formula for $\nabla_{\dot{x}} w$, to the value of $\nabla_{\dot{x}} w$ at $s = 0$. As $\dot{\zeta}(t)$ is the Exp-preimage of the limit, w such that $\nabla_{\dot{x}} w = \pm Q^{-1/2}\phi w$ for $\dot{x} = x_s$ is, in case (II), uniquely determined by $(\nabla_{\dot{x}} w)(0) = \dot{\zeta}(t)$ (for fixed t). Replacing w by $\tilde{w} = Jw$ now amounts to using $J\dot{\zeta}(t)$ instead of $\dot{\zeta}(t)$, that is, to multiplying the Exp-preimage of w by i in the complex vector bundle $\mathcal{H}^N$, and so $\Theta : \mathcal{H}_* \to \mathcal{H}_x$ is complex-linear also in case (II).

Finally, relation $\dot{x} = (\operatorname{sgn} a)v/|v|$ in Lemma 8.3(i), for $x(s) = \operatorname{Exp}(y, sz/|z|)$ at $s = |z|$, shows that Θ sends $z/|z|$, treated as a vertical vector in $T_{(y,z)}\mathcal{L}$, onto $(\operatorname{sgn} a)v(x)/|v(x)|$. Multiplying both vectors by $a|z|$, we obtain our assertion about the Θ-image of $v^N(y,z)$. Also, since $z \in (T_y N)^\perp$, (7.1) gives $\nabla_z u = iaz$. Now $\Theta(u^N(y,z)) = u(x)$ by Lemma 10.1, since on the normal space $\mathcal{L}_y \subset T_y M$ (identified, as usual, with $\{y\} \times \mathcal{L}_y \subset \mathcal{L}$), the normal exponential mapping of N coincides with $\exp_y$. This completes the proof. ∎

§16. A global classification of special Kähler-Ricci potentials

We will now show that every triple (M, g, τ) in which τ is a special Kähler-Ricci potential on a compact Kähler manifold (M, g) is biholomorphically isometric to one of Class 1 or 2 examples, constructed in §5 and §6. Our result (Theorem 16.3) thus provides a complete classification of such triples (M, g, τ).

Lemma 16.1. *Let (S, γ) and (M, g) be complete Riemannian manifolds with open subsets $S' \subset S$ and $M' \subset M$ such that both $S \smallsetminus S'$ and $M \smallsetminus M'$ are unions of finitely many compact submanifolds of codimensions greater than one. Any isometry Ψ of (S', γ) onto (M', g) then can be uniquely extended to an isometry of (S, γ) onto (M, g). If, in addition, (S, γ) and (M, g) are Kähler manifolds and the isometry $\Psi : S' \to M'$ is a biholomorphism, then so is the extension $S \to M$.*

In fact, by the codimension hypothesis S' (or, M') is connected and dense in S (or, in M), and the inclusion mappings $S' \to S$, $M' \to M$ are distance-preserving. As metric spaces, S, M thus are the completions of S' and M'. Our claim now follows since distance-preserving mappings are C^∞ Riemannian isometries ([**16**]; cf. [**12**], Th. 3.10, Ch. IV), with the Kähler case obvious from continuity of J. ∎

Remark 16.2. Given locally trivial fibre bundles $M, \hat{M}$ over a manifold N and a C^∞ diffeomorphism $\Phi : M \to \hat{M}$ sending each fibre M_y onto $\hat{M}_y$, let $\mathcal{H}, \hat{\mathcal{H}}$ be vector subbundles of TM and $T\hat{M}$ with $TM = \mathcal{H} \oplus \mathcal{V}$ and $T\hat{M} = \hat{\mathcal{H}} \oplus \hat{\mathcal{V}}$, where

$\mathcal{V}, \hat{\mathcal{V}}$ are the vertical distributions. If $\varPhi$ sends $\mathcal{H}$ onto $\hat{\mathcal{H}}$, then its differential $d\varPhi_x$ at any $x \in M$, restricted to $\mathcal{H}_x$, preserves any fibre metric or complex vector-bundle structure obtained in both $\mathcal{H}$ and $\hat{\mathcal{H}}$ by lifting a fixed analogous object from TN. (In fact, $d\varPhi_x$ becomes the identity mapping if one uses the differentials of the bundle projections to identify both $\mathcal{H}_x$ and $\hat{\mathcal{H}}_{\varPhi(x)}$ with $T_{\pi(x)}N$.)

Theorem 16.3. *Let τ be a special Kähler-Ricci potential on a compact Kähler manifold (M, g) with $\dim_{\mathbf{C}} M = m \ge 1$. Then, up to a biholomorphic isometry, the triple (M, g, τ) belongs to one of Classes 1 and 2 described in §5 and §6.*

Proof. Let N, N^* be the two critical manifolds of τ, ordered so that either

(1) both N and N^* are of complex dimension $m-1$, or
(2) $N = \{y\}$ for some $y \in M$, while $m \ge 2$ and $\dim_{\mathbf{C}} N^* = m-1$.

(Cf. Proposition 11.5(i).) We will now exhibit ingredients needed to construct a Class 1 example (case (1)) or a Class 2 example (case (2)). First, in both cases, we set $m = \dim_{\mathbf{C}} M$, choose $[\tau_{\min}, \tau_{\max}] \ni \tau \mapsto Q$ to be the assignment associated with M, g, τ as in Proposition 11.5(ii), define τ_0 to be the endpoint of $[\tau_{\min}, \tau_{\max}]$ which is the constant value of τ on N, let c, ε, a be the constants determined by M, g, τ and N as in Lemma 3.1 and Proposition 7.3, and select a positive function $\tau \mapsto r$ on $(\tau_{\min}, \tau_{\max})$ with $dr/d\tau = ar/Q$. Thus, c is left undefined when $\varepsilon = 0$, and, in case (2), $\tau_0 = c$ by Lemma 7.5. Next, in case (1), we let N, h and $\mathcal{L}$ stand, respectively, for our critical manifold, the metric on N defined in the paragraph preceding Lemma 12.4, and the normal bundle of N, carrying the normal connection ∇^{nrm} (Remark 12.2) along with the Hermitian fibre metric whose real part is g restricted to $\mathcal{L}$. In case (2) we in turn set $V = T_yM$ and let $\langle\,,\rangle$ be the Hermitian inner product in V with $\mathrm{Re}\langle\,,\rangle = g(y)$.

The ingredients just defined in case (1) (or, (2)) satisfy the conditions required in §5 (or, §6). First, in both cases, Proposition 11.5(ii) implies the positivity-and-boundary conditions (5.1), and $dQ/d\tau = 2a$ at $\tau = \tau_0$, since $dQ/d\tau = 2\psi$ by Lemma 9.1, while $\psi = a$ on N by Remark 7.4. That, in case (1), either $\varepsilon = 0$ or $c \notin [\tau_{\min}, \tau_{\max}]$ and $\varepsilon = \operatorname{sgn}(\tau - c) = \pm 1$ for all $\tau \in [\tau_{\min}, \tau_{\max}]$ is in turn clear from Lemmas 3.1(iii) and 7.5(ii). Finally, in case (1), assertions (b), (c) of Lemma 12.4 imply that $\varOmega = -2\varepsilon a \omega^{(h)}$ and h is Einstein unless $m = 2$.

Applied to those ingredients, the construction of §5 (or, §6) now yields a compact Kähler manifold of complex dimension m, which we denote by (S, γ) rather than (M, g), and a special Kähler-Ricci potential on (S, γ), still denoted by τ. On both M and S we then also have a function Q equal to $|\nabla\tau|^2$, where the norm and gradient refer to the respective metric g or γ (cf. (b) in §4).

Let $M' \subset M$ and $S' \subset S$ be the open subsets on which $d\tau \ne 0$. Thus, in both cases (1) and (2), $S' = \mathcal{L} \smallsetminus N$, provided that, in case (2), we identify $\{y\} \times T_yM$ with T_yM and, again, denote by $\mathcal{L}$ the total space of the normal bundle of $N = \{y\}$, containing N as the zero section. (See Remarks 5.2 and 6.1.)

We now define a C^∞ diffeomorphism $\varPsi : S' \to M'$, using the diffeomorphism $(0, \infty) \ni r \mapsto \tau \in (\tau_{\min}, \tau_{\max})$ which is the inverse of our function $\tau \mapsto r$ with $dr/d\tau = ar/Q$ (cf. Remark 5.1), and the diffeomorphism $\tau \mapsto s$ of $(\tau_{\min}, \tau_{\max})$ onto $(0, L)$, for $L = \int_{\tau_{\min}}^{\tau_{\max}} Q^{-1/2}\, d\tau$, characterized by $ds/d\tau = (\operatorname{sgn} a) Q^{-1/2}$ with $s = 0$ at $\tau = \tau_0$. The composite $r \mapsto \tau \mapsto s$ is a diffeomorphism $(0, \infty) \to (0, L)$, and leads to a diffeomorphism $\varPhi : \mathcal{L} \smallsetminus N \to \mathcal{L}'$ that may be described as follows.

Letting s also stand for the norm function of the fibre metric in $\mathcal{L}$, we denote by $\mathcal{L}'$ the open subset of $\mathcal{L}$ given by $0 < s < L$, and let $\varPhi(y,z) = (y, sz/|z|)$ for any $y \in N$ and $z \in \mathcal{L}_y \smallsetminus \{0\}$, with $s \in (0,L)$ depending on $r = |z| \in (0,\infty)$ via our composite assignment $r \mapsto s$. (That $\varPhi$ is a diffeomorphism is clear since, if $(y,w) = \varPhi(y,z)$ and $r = |z| > 0$, then $w = sz/r$ and $z = rw/s$ with r obtained from $s = |w|$ via the inverse diffeomorphism $(0,L) \to (0,\infty)$.) Finally, we set $\varPsi = \operatorname{Exp} \circ \varPhi$, which is a diffeomorphism $S' = \mathcal{L} \smallsetminus N \to M'$ in view of Lemma 13.2.

The diffeomorphism $\varPsi : S' \to M'$ sends $\mathcal{H}^N, v^N, u^N$ in $S' = \mathcal{L} \smallsetminus N$ (defined in §15), the functions τ, Q on S', and the metric γ, onto the analogous objects $\mathcal{H}, v, u, \tau, Q, g$ in M', with $v = \nabla\tau$, $u = Jv$ and $\mathcal{H} = v^\perp \cap u^\perp$. In fact, the norm function $s : \mathcal{L}' \to \mathbf{R}$ corresponds under the normal exponential mapping Exp to the arc-length parameter for normal geodesics emanating from N, also denoted by s, and so our claim, for τ, follows since both functions $s \mapsto \tau$ are solutions to the same initial value problem (by Lemma 8.3(ii) and the last paragraph). The claim about Q is now obvious since the dependence of Q on τ in S' is the same as in M'. That $\varPsi$ maps $\mathcal{H}^N$ onto $\mathcal{H}$ is in turn clear since so does Exp (Lemma 15.1), while $\varPhi$ leaves $\mathcal{H}^N$ invariant (since the norm function is constant along any curve in $\mathcal{L}$ tangent to $\mathcal{H}^N$, and so $\varPhi$ multiplies such a curve by a constant factor). As for v^N and u^N, our claim is immediate from the final clause of Lemma 15.1, since $\varPhi$ obviously leaves u^N invariant, while its differential at any point $(y,z) \in \mathcal{L} \smallsetminus N$ sends $v^N(y,z)$ to r/s times ds/dr times $v^N(\varPhi(y,z))$, with s and ds/dr evaluated at $r = |z|$, for r, s as above. (Note that $|a|r\,ds/dr = Q^{1/2}$ by Lemma 8.3(ii), as $dr/d\tau = ar/Q$, and the factor $|az|/|v(x)|$ in Lemma 15.1 equals $|a|sQ^{-1/2}$.) Finally, $\varPsi^*g = \gamma$, since $\varPsi$ maps γ restricted to $\mathcal{H}^N$ onto g restricted to $\mathcal{H}$ in view of Lemma 15.1 and Remark 16.2, while $g(v,v) = g(u,u) = Q$, $g(u,v) = 0$, $g(v,\mathcal{H}) = g(u,\mathcal{H}) = \{0\}$, and the same equalities hold if one replaces $g, v, u, \mathcal{H}$ with $\gamma, v^N, u^N, \mathcal{H}^N$ (cf. (4.1.i) and the line preceding Remark 2.4).

The isometry $\varPsi$ of (S',γ) onto (M',g) is also holomorphic: its differential is complex-linear both on $\mathcal{H}^N$ and on the distribution in $\mathcal{L} \smallsetminus N$ spanned by v^N and u^N, the former conclusion being immediate from Lemma 15.1 and Remark 16.2, the latter obvious as $u^N = Jv^N$ in $\mathcal{L} \smallsetminus N$ (cf. §15) and $u = Jv$ in M.

We may now use Lemma 16.1, as its assumptions hold for our (S,γ), (M,g), S', M' and $\varPsi$ in view of Proposition 11.5(i). This completes the proof. ■

Remark 16.4. In [**7**], Theorem 18.1, we proved a local classification result for special Kähler-Ricci potentials τ on Kähler manifolds (M,g). Namely, up to a biholomorphism, such g and τ always arise, in a neighborhood of any point with $d\tau \neq 0$, from a specific construction of a *local model*. The construction in question also appears at the beginning of §4 of the present paper.

Using the arguments developed in the preceding sections, one can easily obtain, in every complex dimension $m \geq 1$, an analogous classification theorem valid at any *critical* point y of τ, provided that one suitably modifies the local models. Since the case $m = 1$ is trivial (Example 3.2), we assume from now on that $m \geq 2$.

There are two kinds of such modified local models, depending on the dimension of the critical manifold N of τ, containing y. Namely, the first kind is characterized by $\dim_{\mathbf{C}} N = m - 1$, and the second by $N = \{y\}$. Note that these are the only possibilities allowed by Proposition 7.3.

A modified local model of the first kind is obtained from $\mathcal{I}, \tau, Q, r, a, \varepsilon, c, m, N, h, \mathcal{L}, \mathcal{H}, \langle\,,\rangle$, having the properties listed in the second paragraph of §4, along with a finite endpoint τ_0 of $\mathcal{I}$, satisfying the additional assumptions stated in part (ii) of Lemma 4.4. Our local model then is the triple $(U^{\mathrm{o}}, g, \tau)$ obtained in Lemma 4.4(ii).

The data needed to build a modified local model of the second kind consist of a half-open interval $\mathcal{I}'$, with the endpoint denoted by c, a C^∞ function Q of the variable $\tau \in \mathcal{I}'$, such that $Q = 0$ and $dQ/d\tau = 2a$ at $\tau = c$, for some $a \in \mathbf{R} \smallsetminus \{0\}$, while $Q > 0$ on $\mathcal{I}' \smallsetminus \{c\}$, and, finally, an m-dimensional complex vector space V with a Hermitian inner product $\langle\,,\rangle$. Note that the assumptions made in §6 hold here as well, except for those involving the other endpoint of $\mathcal{I}'$. However, a "one end" part of the construction in §6 still works exactly as before, leading to a local model (U, g, τ) formed by a Kähler metric g with a special Kähler-Ricci potential τ on a neighborhood U of 0 in V. An additonal ingredient of the construction is the choice of a function r on $\mathcal{I}' \smallsetminus \{c\}$ with $dr/d\tau = ar/Q$.

Here is an outline of a classification argument involving the new local models. Assuming that y is a critical point of a special Kähler-Ricci potential τ on a Kähler manifold (M, g) with $\dim_{\mathbf{C}} M = m \geq 2$ and N is the critical manifold of τ containing y, we will now define data $\mathcal{I}, \tau, Q, r, a, \varepsilon, c, m, N, h, \mathcal{L}, \mathcal{H}, \langle\,,\rangle, \tau_0$ (if $\dim_{\mathbf{C}} N = m-1$), or $\mathcal{I}', \tau, Q, r, a, c, m, V, \langle\,,\rangle$ (if $N = \{y\}$), having the required properties. Then we will explain why such a local model is biholomorphically equivalent to the pair (g, τ) on a neighborhood of y. The distinction between the two kinds of data (and models) will be a consequence of the dichotomy established in Proposition 7.3.

First, $m = \dim_{\mathbf{C}} M$ is already defined. Next, let $a \neq 0$ be the constant associated with N as in Proposition 7.3. The restriction of $Q = |\nabla\tau|^2$ to any sufficiently small connected neighborhood U of y is a C^∞ function of τ (Lemma 9.1). Also, the values assumed by τ on U form a half-open interval $\mathcal{I}'$ with the endpoint at $\tau_0 = \tau(y)$. The last fact follows since τ is a Morse-Bott function and its Hessian at any critical point is semidefinite (See Example 11.1, Proposition 7.3 and Lemma 11.2.)

Our Q thus becomes a C^∞ function of the variable $\tau \in \mathcal{I}'$, with $dQ/d\tau = 2a$ at $\tau = \tau_0$ (see Lemma 9.1 and Remark 7.4). Let us now fix a positive function r of $\tau \in \mathcal{I}' \smallsetminus \{\tau_0\}$ with $dr/d\tau = ar/Q$, and choose $c, \varepsilon \in \mathbf{R}$ as in Lemma 3.1, so that c is defined only when $\varepsilon = \pm 1$.

According to Proposition 7.3, two cases are possible. First, it may happen that $\dim_{\mathbf{C}} N = m - 1$. We then choose $\mathcal{L}$ to be the normal bundle of N, and let h be the metric on N defined in the paragraph preceding Lemma 12.4 (that is, a specific multiple of the submanifold metric of N), while a connection and a parallel Hermitian fibre metric in $\mathcal{L}$ are selected as in the paragraph following (1), (2) in the proof of Theorem 16.3.

In the remaining case $N = \{y\}$, we set $V = T_yM$ and choose $\langle\,,\rangle$ so that $\mathrm{Re}\langle\,,\rangle = g(y)$. Note that Lemma 7.5(ii) then gives $\tau_0 = c$.

The data just defined thus satisfy, in the former case, all the conditions required in Lemma 4.4(ii) (cf. (b), (c) in Lemma 12.4), and, in the latter, all the assumptions of a "one end" version of §6. This allows us, in either case, to construct the corresponding local model.

The biholomorphism between the local model in question and a neighborhood of

y in M is $\varPsi = \operatorname{Exp} \circ \varPhi$, defined as in the proof of Theorem 16.3; here, however, instead of Lemma 13.2 we simply use the inverse mapping theorem. The rest of the proof is an exact replica of the argument that we used to establish Theorem 16.3.

§17. The conformally-Einstein case: six types

This section introduces a systematic case-by-case approach to classifying compact Kähler manifolds which are almost-everywhere conformally Einstein. We first divide them into six disjoint types (a1), (a2), (b1), (b2), (c1), (c2), and then prove, in Theorem 17.4, that three of the six types are, in fact, empty. To define the types, we distinguish three local cases. Namely, in Proposition 11.5 we showed that, whenever τ is a special Kähler-Ricci potential on a compact Kähler manifold, $Q = |\nabla\tau|^2$ is a C^∞ function of τ. However, in the almost-everywhere conformally Einstein case, that is, when (1.1) is replaced by the stronger assumption (1.2) (in complex dimensions $m \geq 3$) or (1.3) (for $m = 2$), the same conclusion holds even without compactness, and in addition the function $\tau \mapsto Q$ is *rational*. This result, established in [**7**], Proposition 22.1, is stated below; note that, by (1.4), our assumptions imply (1.1), and so Lemma 3.1 may be applied. (In [**7**], ϕ was used instead of ε, which makes no difference, since ϕ and ε are either both zero or both nonzero.)

Proposition 17.1. *If* M, g, m, τ *satisfy* (1.2) *with* $m \geq 3$ *or* (1.3) *with* $m = 2$, *then, for* ε *and* c *as in* Lemma 3.1, *one of the following three cases occurs*:

(a) $\varepsilon = 0$,
(b) $\varepsilon \neq 0$ *and* $c = 0$,
(c) $\varepsilon \neq 0$ *and* $c \neq 0$.

In all three cases, $Q = |\nabla\tau|^2 : M \to \mathbf{R}$ *is a rational function of* τ. *Namely,*

(i) $Q = -K\tau^2 + (2m-1)^{-1}[\alpha\tau^{2m-1} - \eta/m]$ *in case* (a),
(ii) $Q = m^{-1}K\tau + \alpha\tau^{m+1} - 2(m+1)^{-1}\eta/m$ *in case* (b),

for some constants K, α, η. *In case* (c), *there exist constants* A, B, C *such that*

$$Q = (t-1)[A + BE(t) + CF(t)] \quad \textit{for} \quad t = \tau/c,$$

with $F(t) = (t-1)^{-m}(t-2)t^{2m-1}$, $E(t) = (t-1)\sum_{k=1}^{m} \frac{k}{m}\binom{2m-k-1}{m-1}t^{k-1}$, *and* $\tau \neq c$ *everywhere in* M *unless* $C = 0$. ■

Given a quadruple (M, g, m, τ) with (1.2) for $m \geq 3$, or (1.3) for $m = 2$, and such that M is compact, we thus have one of conditions (a), (b), (c) in Proposition 17.1. At the same time, τ is a special Kähler-Ricci potential (cf. (1.4)). Hence, by Proposition 11.5, each of the cases (a), (b), (c) leads to two subcases: (a1), (a2), or (b1), (b2), or (c1), (c2), corresponding to (1) and (2) in Proposition 11.5(i).

As a result, every quadruple (M, g, m, τ) with the stated properties belongs to exactly one of the six *types* (a1), (a2), (b1), (b2), (c1), (c2) just described.

To show that three of the six types are empty, we first prove two lemmas.

Lemma 17.2. *Let* $f = (k-1)\beta^{k+1} - (k+1)\beta^k + (k+1)\beta - (k-1)$ *for* $\beta \in \mathbf{R}$ *and an integer* $k \geq 2$. *Then* $f \neq 0$ *unless* $\beta = 1$ *or* $\beta = (-1)^k$.

In fact, $d^2f/d\beta^2 = k(k+1)(k-1)(\beta-1)\beta^{k-2}$, and so $f' = df/d\beta$ is strictly decreasing (or, increasing) on $(0,1)$ (or, on $(1,\infty)$), while $(-1)^k f'$ is strictly decreasing on $(-\infty,0)$. Evaluating f' at 1, 0 and -1, we now obtain $f'>0$ on $(0,1)\cup(1,\infty)$, and, if k is even, $f'>0$ on $(-\infty,0]$ while, if k is odd, $f'<0$ on $(-\infty,\beta_0)$ and $f'>0$ on $(\beta_0,0)$, for some $\beta_0\in(-1,0)$. Therefore, evaluating f at 1, 0 and -1, we see that $f>0$ on $(-\infty,-1)$ and $f<0$ on $(-1,1)$ (for odd k), $f<0$ on $(-\infty,1)$ (for even k), and $f>0$ on $(1,\infty)$ (for all integers $k\ge 2$). ■

Another, purely algebraic proof of Lemma 17.2 can be obtained by noting that f equals $(\beta-1)^3\Pi(\beta)$ with $\Pi(\beta)=\sum_{j=1}^{k-1} j\,(k-j)\beta^{j-1}$, while $\Pi(\beta)/(\beta+1)$ or $\Pi(\beta)$ is a sum of squares, as $\Pi(\beta)=2^{2-k}\sum_{1\,\le\, j\,\le\, k/2} j\binom{k+1}{2j+1}(\beta+1)^{k-2j}(\beta-1)^{2j-2}$.

Lemma 17.3. *Let the positivity-and-boundary conditions* (5.1) *hold for some given* $\tau_{\min},\tau_{\max}$ *and a function* $\tau\mapsto Q$ *defined by the formula in* (i) *or* (ii) *of* Proposition 17.1*, with an integer* $m\ge 2$ *and real constants* K,α,η.

(a) *In case* (i) *we have* $Q=K(\tau_0^2-\tau^2)$ *and* $\tau_{\max}=-\tau_{\min}=|\tau_0|$ *for some* $\tau_0\ne 0$*, while* $\alpha=0$*,* $K>0$ *and* $\eta<0$.

(b) *In case* (ii)*,* $\tau_{\max}=-\tau_{\min}>0$.

Proof. Since $\tau_{\min}\ne\tau_{\max}$, we may write $\{\tau_{\min},\tau_{\max}\}=\{\tau_0,\tau_1\}$ with $\tau_0\ne 0$. Let φ_0 and φ_1 be the values at τ_0 and τ_1 of any function φ of the variable τ, such as Q or ψ given by $2\psi=dQ/d\tau$. Also, let $k=2m-2$ and $k'=2m-1$ (case (i)), or $k=m$ and $k'=1$ (case (ii)). For $\beta=\tau_1/\tau_0$ and $f=f(\beta)$ as in Lemma 17.2, assuming (i) or (ii) we get $2\tau_0^{-k-1}[Q_0-Q_1+(\tau_1-\tau_0)(\psi_0+\psi_1)]=\alpha f(\beta)/k'$. This gives $\alpha f(\beta)=0$, since, by (5.1), $Q_0=Q_1=\psi_0+\psi_1=0$. As $\tau_1\ne\tau_0$ (that is, $\beta\ne 1$), Lemma 17.2 now implies that $\alpha=0$, or k is odd and $\tau_1=-\tau_0$.

In case (i), $k=2m-2$ is even, and so $\alpha=0$, while (i) with $\alpha=0$ and $Q_0=Q_1=0$ easily yields (a). (In both cases, $|K|+|\alpha|>0$ due to the nonzero-derivative requirement in (5.1).) In case (ii), however, $\alpha\ne 0$, since relation $\alpha=0$ in (ii), along with $Q_0=Q_1=0$ and $\tau_1\ne\tau_0$, would give $K=\eta=0$. Thus, $\tau_1=-\tau_0\ne 0$ in case (ii), and (b) follows, completing the proof. ■

Theorem 17.4. *Let* M,g,m,τ *satisfy* (1.2) *with* $m\ge 3$ *or* (1.3) *with* $m=2$*. If* M *is compact, then* (M,g,m,τ) *cannot be of type* (a2)*,* (b1) *or* (b2).

In fact, type (a2) is empty by Lemma 7.5(i). Next, if (M,g,m,τ) were of type (b1) or (b2), Lemma 3.1 would give $\tau\ne c$, that is, $\tau\ne 0$, everywhere in the open set $M'\subset M$ on which $d\tau\ne 0$. As M' is connected and dense in M (Remark 2.3(ii)), τ would be nonnegative or nonpositive on M, even though $\tau_{\max}=-\tau_{\min}>0$ by Lemma 17.3(b) (which applies in view of Propositions 17.1(ii) and 11.5(ii)). ■

§18. Type (a1): examples and a classification theorem

The simplest (and well known) examples of quadruples (M,g,m,τ) with compact M, satisfying (1.2), are certain Riemannian products having S^2 as a factor; see [**7**], §25. This section begins with a slightly more general construction of such examples, in which g is a locally reducible metric on the compact total space M of an S^2 bundle with a flat connection. We then show (Theorem 18.1) that, up to τ-preserving biholomorphic isometries, the quadruples with (1.3) constructed here are precisely the quadruples of type (a1) described in §17. In other words, Theorem 18.1 provides a complete classification for type (a1).

First. let there be given an integer $m \geq 2$, a real number $K > 0$, a compact Kähler-Einstein manifold (N,h) of complex dimension $m-1$ with the Ricci tensor $\mathrm{r}^{(h)} = (3-2m)Kh$, and a C^∞ complex line bundle $\mathcal{L}$ over N with a Hermitian fibre metric and a fixed flat connection making the metric parallel (that is, a flat $\mathrm{U}(1)$ connection). The simplest choice of such $\mathcal{L}$ is the product bundle $\mathcal{L} = N \times \mathbf{C}$.

Let $\mathcal{E} = N \times \mathbf{R}$ now denote the product real-line bundle over N, with the obvious "constant" Riemannian fibre metric, and let M be the unit-sphere bundle of the direct sum $\mathcal{L} \oplus \mathcal{E}$. Thus, M is a 2-sphere bundle over N, with $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V}$ is the vertical distribution (tangent to the fibres), and $\mathcal{H}$ is the restriction to M of the horizontal distribution of the direct-sum connection in $\mathcal{L} \oplus \mathcal{E}$. Since the latter connection is flat, the distributions $\mathcal{V}, \mathcal{H}$ are both integrable. We now define a metric g on M by choosing g on $\mathcal{V}$ to be $1/K$ times the standard unit-sphere metric of each fibre, declaring $\mathcal{V}$ and $\mathcal{H}$ to be g-orthogonal, and letting g on $\mathcal{H}$ be the pullback of h under the bundle projection $M \to N$. Finally, we define $\tau : M \to \mathbf{R}$ to be any nonzero constant times the restriction to M of the composite $\mathcal{L} \oplus \mathcal{E} \to \mathcal{E} \to \mathbf{R}$ consisting of the direct-sum projection morphism $\mathcal{L} \oplus \mathcal{E} \to \mathcal{E}$ followed by the Cartesian-product projection $\mathcal{E} = N \times \mathbf{R} \to \mathbf{R}$.

The flat connection in $\mathcal{L} \oplus \mathcal{E}$ allows us to treat it, locally, as a product bundle, which makes (M,g), locally, a Riemannian-and-Kähler product of (U,h) and the (oriented) sphere S^2 with a metric of constant curvature K.

Theorem 18.1. *Every quadruple (M,g,m,τ) constructed as above satisfies condition* (1.3)*, and belongs to type* (a1) *defined in* §17.

Conversely, every quadruple (M,g,m,τ) with compact M which satisfies (1.2) *with $m \geq 3$ or* (1.3) *with $m = 2$, and belongs to type* (a1) *is, up to a τ-preserving biholomorphic isometry, obtained from the above construction.*

We will prove Theorem 18.1 at the end of this section. First, let us define the *Riemann sphere* S of a complex vector space V with $\dim_{\mathbf{C}} V = 1$ to be, as usual, the complex projective line of all complex lines through 0 in $V \times \mathbf{C}$. Thus, S contains V as an open subset: $V = S \smallsetminus \{\infty\}$, where ∞ is the complex line $V \times \{0\}$ viewed as an element of S.

If, in addition, V is endowed with a Hermitian inner product $\langle\,,\rangle$, while $K > 0$ and $\tau_0 \neq 0$ are real constants, we use the constant $a = -K\tau_0$, the norm function $r : V \to [0,\infty)$ of $\langle\,,\rangle$, and functions τ, Q of the variable $r \in (0,\infty)$ chosen so that $ar\,d\tau/dr = Q = K(\tau_0^2 - \tau^2)$ and $|\tau| < |\tau_0|$, to define the Riemannian metric $(ar)^{-2}Q\,\mathrm{Re}\langle\,,\rangle$ on $V \smallsetminus \{0\}$, where $\mathrm{Re}\langle\,,\rangle$ is the Euclidean metric.

Lemma 18.2. *For any $V, \langle\,,\rangle, K, \tau_0, \tau$ and Q with the properties just listed, the resulting metric $(ar)^{-2}Q\,\mathrm{Re}\langle\,,\rangle$ on $V \smallsetminus \{0\}$ has a C^∞ extension to a metric γ of constant Gaussian curvature K on the Riemann sphere S.*

Furthermore, an isometry $\varphi : S \to S_1$ between (S,γ) and the unit sphere S_1 about $(0,0)$ in $V \times \mathbf{R}$ with $1/K$ times its submanifold metric can be defined by $\varphi(z) = \chi(z)/|\chi(z)|$ for $z \in V$ and $\varphi(\infty) = (0,-1)$, where $\chi(0) = (0, \sqrt{K}\tau_0)$ and $\chi(z) = (\sqrt{Q}z/|z|, \sqrt{K}\tau)$ if $z \in V \smallsetminus \{0\}$.

Proof. The extension γ of our metric to S exists since this is a special case of the construction in §6, for $m = 1$ and $Q = K(\tau_0^2 - \tau^2)$ on the interval $[\tau_{\min}, \tau_{\max}]$ with $\tau_{\min} = -|\tau_0|$ and $\tau_{\max} = |\tau_0|$.

That $\varphi : S \to S_1$ is an isometry will follow from Lemma 16.1 once we have shown that our χ maps S bijectively onto the sphere $\hat{S} \subset V \times \mathbf{R}$ of radius $\sqrt{K}|\tau_0|$ centered at 0 and, on $V \smallsetminus \{0\}$, the pullback under χ of the submanifold metric of $\hat{S}$ equals Q/r^2 times the Euclidean metric $\mathrm{Re}\langle\,,\rangle$ on V. This is in turn clear since χ sends *lines* (through 0 in V) and *circles* (about 0, in V) into *meridians* and, respectively, *parallels* in $\hat{S}$, in the cartographic terminology based on the *poles* $(0, \pm\sqrt{K}\tau_0)$. Our lines are orthogonal to circles, and meridians to parallels; thus, all we need to show is that χ restricted to any line or circle deforms the Euclidean metric by the conformal factor Q/r^2.

For the circles, this is immediate: the obvious S^1 actions make χ equivariant, and χ sends the circle of any radius $r > 0$ onto a parallel which is a circle of radius $\sqrt{Q}$, where $Q = K(\tau_0^2 - \tau^2)$, with the required ratio $\sqrt{Q/r^2}$ of the radii.

For the lines, let $z(r) = rz_0$ with $\langle z_0, z_0\rangle = 1$. Then $\chi(z(r)) = (\sqrt{Q}\,z_0, \sqrt{K}\tau)$ (where τ, Q depend on $r \in (0,\infty)$ as before), and so, as $dQ/d\tau = -2K\tau$, we have $|d[\chi(z(r))]/dr|^2 = K(Q + K\tau^2)(d\tau/dr)^2/Q$. Hence $|d[\chi(z(r))]/dr|^2$ equals Q/r^2, that is, Q/r^2 times $|d[z(r)]/dr|^2$. (In fact, $Q + K\tau^2 = K\tau_0^2$, $a = -K\tau_0$ and, as $dr/d\tau = ar/Q$, we have $d\tau/dr = Q/(ar)$.) This completes the proof. ■

Proof of Theorem 18.1. For (M, g, m, τ) obtained as above, (1.3) is immediate from [**7**], §25. (In [**7**], g is a product metric on a product bundle; however, the argument is local, and so the conclusion holds in the local-product case as well.) Since τ is constant in the N-direction of a local product decomposition for g, both $v = \nabla\tau$ and $u = Jv$ are tangent to the S^2 factor, and so the horizontal distribution $\mathcal{H}$ equals $v^\perp \cap u^\perp$ on the set $M' \subset M$ on which $d\tau \neq 0$. As $\mathcal{H}$ is integrable, Lemma 12.1(iv) gives $\phi = 0$ on M', that is, $\varepsilon = 0$ (cf. Lemma 3.1(iii)). Consequently, (M, g, m, τ) is of type (a1), since type (a2) is excluded by Theorem 17.4.

Conversely, let a quadruple (M, g, m, τ) with compact M satisfy (1.2) with $m \geq 3$, or (1.3) with $m = 2$, and be of type (a1). By (1.4), τ is a special Kähler-Ricci potential, and so, according to Theorem 16.3, (M, g, τ) is a Class 1 triple, obtained as in §5. (It cannot belong to Class 2, as τ would then have a one-point critical manifold, cf. Remark 6.1, contrary to the definition of type (a1).)

As (M, g, τ) is a Class 1 triple, the assignment $\tau \mapsto Q$ obtained in Proposition 11.5(ii) and the distribution $\mathcal{H} = v^\perp \cap u^\perp$, with v, u as above, coincide with objects used in the construction of §5: the function $\tau \mapsto Q$, and the horizontal distribution of the connection in the line bundle $\mathcal{L}$. (Cf. (b) in §4.) The definition of type (a1) gives $\varepsilon = 0$, and hence $\phi = 0$ on M' (see Lemma 3.1(iii)). Thus, $\mathcal{H}$ is integrable by Lemma 12.1(iv), and the connection in $\mathcal{L}$ is flat.

By Proposition 17.1(i) and Lemma 17.3(a), $Q = K(\tau_0^2 - \tau^2)$ and $\tau_{\max} = -\tau_{\min} = |\tau_0|$ for some $K \in (0,\infty)$ and $\tau_0 \in \mathbf{R} \smallsetminus \{0\}$. This K and $N, h, \mathcal{L}, \mathcal{H}, \langle\,,\rangle$ used in the construction of §5 lead to an S^2 bundle defined at the beginning of the present section, which we now denote by $\hat{M}$, with a Kähler metric $\hat{g}$, a special Kähler-Ricci potential $\hat{\tau}$, and two distributions $\hat{\mathcal{V}}$ and $\hat{\mathcal{H}}$. Note that the construction leaves us the freedom to multiply $\hat{\tau}$ by a nonzero constant.

Let Φ now be the fibre-preserving C^∞ diffeomorphism of the $\mathbf{C}\mathrm{P}^1$ bundle M over N onto the S^2 bundle $\hat{M}$ over N which operates between the fibres over each $y \in N$ as the canonical isometry φ defined in Lemma 18.2. Since φ is also orientation-preserving (for the obvious 2-sphere orientations), it is holomorphic, that is, Φ maps fibres of M biholomorphically onto those of $\hat{M}$.

By Lemma 18.2, τ corresponds under Φ to a constant multiple of $\hat{\tau}$ (as the $\mathbf{R}$-component of χ is $\sqrt{K}\tau$), and Φ preserves horizontality of curves, that is, sends the distribution $\mathcal{H}$ onto $\hat{\mathcal{H}}$. Therefore, Φ is a holomorphic isometry: we just verified that for the restriction of Φ to the fibres, while the differential of Φ at any point, restricted to $\mathcal{H}$, is complex-linear and isometric by Remark 16.2. Finally, both $\Phi^*\hat{g}$ and g make $\mathcal{H}$ orthogonal to the vertical distribution in M, which completes the proof. ∎

§19. A STRUCTURE THEOREM FOR TYPE (c1)

We now proceed to establish a structure theorem for type (c1) of §17, which reduces its classification to the question of finding all objects that satisfy conditions (19.1) – (19.4) below with $1 \notin I$. The latter question is addressed in [**8**]; for a brief summary of the results of [**8**], see §1.

We conclude this section with two corollaries concerning type (c2). In [**8**] they are used to show that type (c2) is actually empty.

Let a sextuple m, I, Q, A, B, C consist of

(19.1) an integer $m \geq 2$, a nontrivial closed interval $I \subset \mathbf{R}$, constants A, B, C, and the function $Q = (t-1)[A + BE(t) + CF(t)]$ of the variable $t \in I$,

with E, F as in Proposition 17.1. We then consider the following conditions:

(19.2)
a) Q is analytic on I, that is, I does not contain 1 unless $C = 0$.
b) $Q = 0$ at both endpoints of I.
c) $Q > 0$ at all interior points of I.
d) dQ/dt is nonzero at both endpoints of I.
e) The values of dQ/dt at the endpoints of I are mutually opposite.

Lemma 19.1. *Let* (M, g, m, τ) *be a quadruple satisfying* (1.2) *with* $m \geq 3$ *or* (1.3) *with* $m = 2$*, and such that* M *is compact, while* $\varepsilon c \neq 0$ *for* ε, c *as in* Lemma 3.1. *If the assignment* $[\tau_{\min}, \tau_{\max}] \ni \tau \mapsto Q \in \mathbf{R}$ *of* Proposition 11.5(ii)*, cf.* (1.4)*, is treated as a* C^∞ *function of the variable* $t = \tau/c$*, then one has* (19.1) *and* (19.2) *for these* m, Q *along with* $I = [\tau_{\min}/c, \tau_{\max}/c]$ *and some* A, B, C.

In fact, Proposition 17.1 yields (19.1); hence Q is a rational function of t, so that its C^∞-differentiability on I amounts to analyticity, and (19.2) follows from Proposition 11.5(ii). ∎

Given m, I, Q, A, B, C with (19.1) – (19.2) and with $1 \notin I$, let us choose

(19.3) $a, c \in \mathbf{R} \smallsetminus \{0\}$ and $\varepsilon = \pm 1$ such that $\pm 2ac$ are the values of dQ/dt at the endpoints of I, and $\varepsilon c(t-1) > 0$ for all $t \in I$.

(Such a, c, ε exist by (19.2.d) and (19.2.e), since $1 \notin I$.) Let us also select

(19.4) a compact Kähler-Einstein manifold (N, h) with $\dim_{\mathbf{C}} N = m - 1$ having the Ricci tensor $\mathrm{r}^{(h)} = \kappa h$ for $\kappa = \varepsilon m A/c$, a C^∞ complex line bundle $\mathcal{L}$ over N, and a connection in $\mathcal{L}$ making a Hermitian fibre metric $\langle\,,\rangle$ parallel, with the curvature form $\Omega = -2\varepsilon a \omega^{(h)}$, where $\omega^{(h)}$ is the Kähler form of (N, h).

Remark 19.2. The existence of $\mathcal{L}$ with the connection required in (19.4) is by no means guaranteed for a given choice of data (19.1) – (19.3) with $1 \notin I$ and (N,h) as in (19.4). For instance, m, I, Q, A, B, C then must satisfy the following additional condition: *either* $A = 0$, *or the values of* $A^{-1}dQ/dt$ *at the endpoints of* I *are rational.* In fact, by (19.3) – (19.4) with $A \neq 0$, those values are $\pm m/2$ times the ratio $c_1(\mathcal{L})/c_1(N)$ of two integral classes in $H^2(N, \mathbf{R})$. Further necessary conditions follow from a theorem of Kobayashi and Ochiai [**13**]. See [**8**], end of §39, for details.

We will now use any given data with (19.1) – (19.4) and $1 \notin I$ to construct a quadruple (M, g, m, τ) with (1.3), belonging to type (c1) of §17, and such that M is a holomorphic $\mathbf{C}\mathrm{P}^1$ bundle over N.

First, we choose a positive function r of the variable t restricted to the interior of I, such that $dr/dt = acr/Q$, with Q depending on t as in (19.1). Now $dr/d\tau = ar/Q$, while (19.2) gives (5.1), for Q, r treated as functions of the variable $\tau = ct$ in the interval $[\tau_{\min}, \tau_{\max}] = cI$ or $(\tau_{\min}, \tau_{\max})$. By Remark 5.1, r ranges over $(0, \infty)$, and so τ, Q restricted to the interior of I become functions of $r \in (0, \infty)$.

We will also use the symbol r for the norm function $\mathcal{L} \to (0, \infty)$ of $\langle\,,\rangle$. Being functions of $r > 0$, both τ and Q thus may now be viewed as functions on $\mathcal{L} \smallsetminus N$. Let g now be the metric on the complex manifold $\mathcal{L} \smallsetminus N$ such that the vertical subbundle $\mathcal{V}$ of the tangent bundle is g-orthogonal to the horizontal distribution $\mathcal{H}$ of the connection chosen in $\mathcal{L}$, while g on $\mathcal{H}$ equals $2|\tau - c|$ times the pullback of h to $\mathcal{H}$ under the bundle projection $\mathcal{L} \to N$, and g on $\mathcal{V}$ is $Q/(ar)^2$ times the standard Euclidean metric $\mathrm{Re}\,\langle\,,\rangle$.

This is clearly a special case of the construction in §5. Consequently, g and τ have C^∞ extensions to a metric and a function, still denoted by g, τ, on the compact complex manifold M obtained as the projective compactification of $\mathcal{L}$.

Theorem 19.3. *Let* M, g, m, τ *be obtained via the above construction from some data with* (19.1) – (19.4) *such that* $1 \notin I$. *Then* M *is compact, the quadruple* (M, g, m, τ) *satisfies* (1.3), *and* (M, g, m, τ) *belongs to type* (c1) *of* §17.

Conversely, if (M, g, m, τ) *with compact* M *satisfies* (1.2) *with* $m \geq 3$ *or* (1.3) *with* $m = 2$ *and is of type* (c1), *then, up to a* τ*-preserving biholomorphic isometry, it is obtained as above from* m *with some data* (19.1) – (19.4) *such that* $1 \notin I$.

Proof. According to [**7**], Proposition 23.3, M, g, m, τ constructed above satisfy (1.3), since our description of g and τ on $\mathcal{L} \smallsetminus N$ is a special case of that in [**7**], §23, case (iii). In addition, since $\varepsilon = \pm 1$, assertion (c) in [**7**], §16 states that $\phi \neq 0$ and our constant $c \neq 0$ is the same as in Lemma 3.1, and so, by Remark 5.2, (M, g, m, τ) is of type (c1). This proves the first claim.

Now let (M, g, m, τ) be as in the second claim. By (1.4) and Theorem 16.3, the triple (M, g, τ) belongs to Class 1 or Class 2. It cannot, however, be in Class 2, or have $1 \in I$ for I as in Lemma 19.1, as either condition would imply that τ has a one-point critical manifold (cf. Remark 6.1 and Lemma 7.5), thus contradicting the definition of type (c1). Hence M, g, τ are obtained as in §5 from some data that include a function Q of τ and a Kähler manifold (N, h) with the Ricci tensor $\mathrm{r}^{(h)} = \kappa h$ for some function $\kappa : N \to \mathbf{R}$. As a function on M this Q equals $|\nabla\tau|^2$ (see (b) in §4), while, by Proposition 17.1, $Q = (t-1)[A + BE(t) + CF(t)]$ with $t = \tau/c$. (We use c and ε defined in Lemma 3.1.) Finally, as shown in [**7**], Remarks 23.2 and 9.4, κ is the constant $\varepsilon m A/c$. This completes the proof. ∎

Lemma 19.1 and Theorem 19.3 also lead to conclusions about type (c2):

Corollary 19.4. *Let* M, g, m, τ *with compact* M *satisfy* (1.2) *with* $m \geq 3$*, or* (1.3) *with* $m = 2$*, and belong to type* (c2) *of* §17*. Then conditions* (19.1) *and* (19.2) *hold for* m *and some* I, Q, A, B, C *such that* $1 \in I$*.*

This is clear since, for I, Q, A, B, C chosen as in Lemma 19.1, $[\tau_{\min}, \tau_{\max}]$ contains $\tau = c$ (by Lemma 7.5(ii)), and so I contains the point $t = 1$. ■

Corollary 19.5. *One of the following two assertions holds for any given quadruple* (M, g, m, τ) *with* (1.2) *and* $m \geq 3$*, or* (1.3) *and* $m = 2$*, and with compact* M*.*

(i) *Up to a* τ*-preserving biholomorphic isometry,* (M, g, m, τ) *arises from the construction preceding* Theorem 19.3*, or from that in* §18*.*
(ii) *We have* (19.2) *with* (19.1) *for our* m *and some* I, Q, A, B, C *with* $1 \in I$*.*

In fact, (M, g, m, τ) belongs to one of the six types introduced in §17. However, types (a2), (b1), (b2) are excluded by Theorem 17.4, types (a1) and (c1) lead to (i) (see Theorems 18.1 and 19.3), and, for type (c2), Corollary 19.4 gives (ii). ■

As it eventually turns out, type (c2) is empty: as shown in [**8**], Proposition 19.2, the conclusion of Corollary 19.4 and (ii) in Corollary 19.5 cannot occur, since conditions (19.1) – (19.2) with any $m \geq 2$ imply that $1 \notin I$.

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