

THE ETA INVARIANT IN THE DOUBLY KÄHLERIAN CONFORMALLY COMPACT EINSTEIN CASE

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ABSTRACT. On a 3-manifold bounding a compact 4-manifold, let a conformal structure be induced from a complete Einstein metric which conformally compactifies to a Kähler metric. Formulas are derived for the eta invariant of this conformal structure under additional assumptions. One such assumption is that the Kähler metric admits a special Kähler-Ricci potential in the sense defined by Derdzinski and Maschler. Another is that the Kähler metric is part of an ambitoric structure, in the sense defined by Apostolov, Calderbank and Gauduchon, as well as a toric one. The formulas are derived using the Duistermaat-Heckman theorem. This result is closely related to earlier work of Hitchin on the Einstein selfdual case.

1. INTRODUCTION

In [H], Hitchin computed an obstruction, in terms of a bound on the eta invariant for a conformal structure on the 3-sphere to be induced from a complete self-dual Einstein metric on the 4-ball. Following LeBrun [L], such conformal structures are said to have *positive frequency*, drawing on an analogy between this case and the classical obstruction for a smooth function on the circle to be the boundary value of a holomorphic function on the disk.

In this paper we give formulas for the eta invariant of a conformal structure induced from another type of asymptotically hyperbolic Einstein metric in dimension four, namely one that is conformal to a (conformally compact) Kähler metric, and satisfies additional technical assumptions. These formulas are obtained via moment map techniques, specifically the Duistermaat-Heckman Theorem. The assumptions are either that the Kähler metric admits a special Kähler-Ricci potential in the sense of [DM1, DM2], or that it is part of an ambitoric structure, in the sense defined recently by Apostolov et. al. [ACG], as well as a toric one. Common to both cases is the existence of a second, oppositely oriented complex structure, with respect to which one has a second Kähler metric in the conformal class, which plays a major role in the derivation of the formulas.

Section 2 is devoted mainly to a review of the theorem of Hitchin. Section 3 gives an overview of the classification of metrics with a special Kähler-Ricci potential in the case of a manifold with boundary, and also discusses the dual Kähler metric. In Section 4 the two formulas are derived.

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2. PRELIMINARIES

2.1. Conformal compactifications of Einstein metrics. In the following let (M, g_E) be *conformally compact Einstein*, i.e., M is a compact manifold with a nonempty boundary, g_E a complete Einstein metric in the interior of M and there exists a smooth defining function τ for the boundary ∂M (so that $\tau \geq 0, \tau|_{\partial M} = 0, d\tau|_{\partial M} \neq 0$) for which

$$g = \tau^2 g_E \text{ is smooth on } M.$$

The pair (g, τ) is called a *conformal compactification* for g_E . The restriction $\gamma = g|_{\partial M}$ varies with τ within a fixed conformal class $[\gamma]$ on ∂M .

Later we will consider such (M, g_E) with a *Kählerian (conformal) compactification*, i.e. satisfying the additional assumption that g is Kähler in the interior of M . The Kählerian assumption, combined with the more technical assumption (4), which will be described in §3, imply that the function $Q := |\nabla\tau|^2$ is constant on each level set of τ (see [DM1]). A compactification with the latter property may be called *uniform*. It is easier to achieve, in a sense, than the more well-known *geodesic* one, where Q is constant in a collar neighborhood of the boundary: from a given compactification, one must solve a PDE to obtain a geodesic compactification, but only an ODE to obtain a uniform one. On the other hand, the above two assumptions imply the uniformity of the compactification throughout the open and dense set of non-critical τ values, and not just in a collar neighborhood of the boundary.

2.2. The theorem of Hitchin. We recall here the result of Hitchin, emphasizing its applicability, in addition to the four-ball, also to other four-manifolds.

Theorem 1 (Hitchin). *Suppose (g, τ) is a conformal compactification of (M, g_E) , with g_E Einstein and M a manifold with boundary of real dimension four. Then*

$$(1) \quad \eta_\gamma(\partial M) = \frac{1}{12\pi^2} \int_M (|W_-|_g^2 - |W_+|_g^2) \text{vol}_g - \sigma(M),$$

where W_+ (W_-) are the (anti-)self dual parts of the Weyl tensor of g , vol_g its volume form, $\sigma(M)$ the signature of M and $\eta_\gamma(\partial M)$ the eta invariant of the boundary metric γ .

Here the eta invariant is defined as the value at $s = 0$ of the series $\sum_{\lambda \neq 0} (\text{sgn}\lambda) |\lambda|^{-s}$ (which is holomorphic for $\text{Re } s > -1/2$), where the summation is over nonzero eigenvalues of the self-adjoint operator on even forms of ∂M , given by

$$B(\alpha) = (-1)^p (*d - d*)\alpha, \quad \alpha \in \Omega^{2p}(\partial M).$$

This invariant depends only on the conformal class of γ in ∂M .

Proof. The Atiyah, Patodi and Singer [APS] signature formula for a manifold with boundary reads [EGH]

$$\sigma(M) = -\frac{1}{24\pi^2} \int_M \text{tr}(R \wedge R) + \frac{1}{24\pi^2} \int_{\partial M} \text{tr}(\Pi \wedge R) - \eta_\gamma(\partial M),$$

where R is the curvature tensor of M and Π is the second fundamental form of ∂M , considered as a one-form valued in (restrictions to ∂M of) endomorphisms of the tangent bundle of M . In the case at hand, this boundary term actually vanishes. In fact, as g is conformal to the Einstein metric g_E , it satisfies a Ricci-Hessian equation of the form $\nabla d\tau + (\tau/2)r = \sigma g$, with r the Ricci curvature and ∇ the covariant derivative of g , for some function σ . As $\partial M = \{\tau = 0\}$, the second fundamental form of the boundary is the Hessian $\nabla d\tau$. But the Ricci-Hessian equation above yields that on the boundary this Hessian is a multiple of the metric, i.e. that ∂M is totally umbilical. Thus, in an adapted orthonormal frame $\{e_i\}_{i=1}^4$ with $\{e_i\}_{i=1}^3$ tangent to, and e_4 a unit normal to ∂M , $\Pi = k \sum_{i=1}^3 e_i \otimes e_i$ for some constant k , which implies $\text{tr}(\Pi \wedge R) = k(e_1 \wedge R_{14} + e_2 \wedge R_{24} + e_3 \wedge R_{34}) + k(R_{1423} + R_{2431} + R_{3412})e_1 \wedge e_2 \wedge e_3 = k(R_{4123} + R_{4231} + R_{4312})e_1 \wedge e_2 \wedge e_3 = 0$ by the Bianchi identity. The result now follows as $\text{tr}(R \wedge R) = 2(|W_+|_g^2 - |W_-|_g^2) \text{vol}_g$. \square

For the four-ball, $\sigma(M) = 0$, and so Hitchin concludes from (1) that the eta invariant of S^3 is nonpositive for conformal structures induced from self-dual Einstein metrics on B^4 , and zero exactly for the standard structure. More generally, the following obvious general bounds hold:

Corollary 2.1.

$$(2) \quad -\frac{1}{12\pi^2} \int_M |W_+|_g^2 \text{vol}_g - \sigma(M) \leq \eta_\gamma(\partial M) \leq \frac{1}{12\pi^2} \int_M |W_-|_g^2 \text{vol}_g - \sigma(M)$$

with equality holding on the left-hand side in the self-dual case, and on the right-hand side in the anti-selfdual case. Moreover, if the compactification is also Kählerian, and g is not anti-selfdual, then the left hand inequality of (2) can be rewritten as

$$(3) \quad -\frac{1}{288\pi^2} \int_M s_g^2 \text{vol}_g - \sigma(M) \leq \eta_\gamma(\partial M),$$

an inequality that holds because on a Kähler surface, W_+ is completely determined by the scalar curvature and the Kähler form (see for example Derdzinski [D]).

3. KÄHLER-RICCI POTENTIALS ON MANIFOLDS WITH BOUNDARY

3.1. Metrics with a Kähler-Ricci potential. We give here an outline of the analog, for manifolds with boundary, of the main theorem in [DM2] (see Theorem 2 below).

Let (M, J, g, τ) be a quadruple consisting of a compact manifold M with a nonempty boundary, having a defining function τ , and a complex structure J along with a Kähler metric g on the interior of M for which

$$(4) \quad \tau \text{ is a special Kähler-Ricci potential.}$$

This requirement on τ is a technical notion which holds in the conformally Einstein case if either the complex dimension of M is at least three, or else it is two and $d\tau \wedge$

$d\Delta_g\tau = 0$, with Δ_g the g -Laplacian. Metrics satisfying (4) have been classified both locally and on compact manifolds in [DM1] and [DM2]. Specifically, a special Kähler-Ricci potential is a Killing potential on a Kähler manifold, i.e. $J\nabla\tau$ is a Killing vector field, for which the following additional condition holds: at any noncritical point of τ , the subspace $\mathcal{H} = (\text{span}(\nabla\tau, J\nabla\tau))^\perp$ of the tangent space is an eigenspace for both the Hessian $\nabla d\tau$ and the Ricci curvature r .

Suppose the interval of τ -values is $[0, \tau_0]$ and the τ -preimage N of the maximal value τ_0 is a critical manifold. By [DM2, Lemma 7.3], N is a complex submanifold of M which is of complex codimension one, or else consists of a single point. Furthermore, *any* critical manifold is the τ -preimage of an extreme value for τ (see the proof of Proposition 11.4 in [DM2]). As τ is a defining function, the zero level set of τ does not contain any critical points (and is the boundary), so that N is the unique critical manifold of τ .

Special Kähler-Ricci potentials give rise to ingredients that are used to give an explicit construction of a metric \bar{g} which is biholomorphically isometric to g . Namely, the function $|\nabla\tau|^2$ is a composite consisting of τ followed by a smooth function $Q(\tau)$ defined on $[0, \tau_0]$. This follows from Lemma 9.1 of [DM2] along with Proposition 11.5(ii) there, for the interval $[0, \tau_0]$ with the proof changed to the case of a compact manifold with boundary. Adopting again the same proposition, Q satisfies the boundary and positivity conditions

$$(5)^{\text{bdps}} \quad Q(\tau_0) = 0, Q'(\tau_0) \neq 0 \text{ and } Q(\tau) > 0 \text{ for } \tau \in [0, \tau_0).$$

Another ingredient is a constant c , defined as follows: the eigenfunction ϕ of the Hessian $\nabla d\tau$ on the subspace \mathcal{H} can be identically zero, in which case c is undefined, or nowhere zero for τ -noncritical points, and then $c = \tau - Q/(2\phi)$ [DM2, Lemma 3.1]. We note for future reference that

$$(6) \quad c \notin [0, \tau_0] \text{ unless the critical manifold } N \text{ consists of a single point and then } c = \tau_0$$

(see [DM2, Lemma 7.5]). Furthermore, the unique nonzero eigenvalue of $\nabla d\tau$ at each point of N is a constant denoted e [DM2, Proposition 7.3 (denoted a there)]. With Q, c and e one defines a metric \bar{g} on a disc bundle S contained in the total space of the normal bundle \mathcal{L} to N , using a Hermitian fiber metric $\langle \cdot, \cdot \rangle$ whose real part is $g|_{\mathcal{L}}$ and a metric h on N defined as either $g|_{TN}$ if $\phi = 0$, or $(2|\tau_0 - c|)^{-1}g|_{TN}$ if $\phi \neq 0$ (if N consists of a single point $\{y\}$ then S is a disc, $\mathcal{L} = T_yM$ and h is undefined). The connection associated with the fiber metric is the normal connection, and its curvature Ω satisfies

$$(7) \quad \Omega = p\omega^h,$$

where ω^h is the Kähler form of h and the constant p equals $\pm 2e$ (see [DM2, Lemma 12.4]).

Regarding N as the zero section $N \subset \mathcal{L}$, the metric \bar{g} is defined as follows:

$$(8) \quad \text{met}$$

if N is a complex codimension one submanifold:

$$\begin{aligned} \text{i]} \quad \bar{g}|_{\mathcal{H}} &= \pi^*h && \text{if } \phi = 0, \\ \text{ii]} \quad \bar{g}|_{\mathcal{H}} &= 2|\tau - c| \pi^*h && \text{if } \phi \neq 0, \end{aligned}$$

if N consists of a single point:

$$\text{iii]} \quad \bar{g}|_{\mathcal{H}} = (2|\tau - c|/(|e|r^2)) \pi^*h,$$

and in all cases:

$$\bar{g}|_{\mathcal{V}} = (Q(\tau)/(er)^2) \operatorname{Re} \langle \cdot, \cdot \rangle.$$

Here, if N is a complex codimension one submanifold, \mathcal{H}, \mathcal{V} are the horizontal distribution defined by the connection and the vertical distribution, respectively, which are also declared \bar{g} -orthogonal to each other. If N consists of a single point, then \mathcal{V} is the distribution on the vector space $T_y L \setminus 0$ defined at each nonzero vector as the complex line through this vector, while \mathcal{H} is the distribution orthogonal to \mathcal{V} with respect to $\operatorname{Re} \langle \cdot, \cdot \rangle$. Finally, $r = r(\tau)$ is a nonnegative function on $[0, \tau_0]$ which vanishes only at $\tau = \tau_0$ and satisfies the differential equation $dr/d\tau = er/Q$ in $(0, \tau_0)$. Abusing notation by regarding r also as the norm function of $\langle \cdot, \cdot \rangle$, it is a function on \mathcal{L} . Using the inverse map $\tau(r)$ to $r(\tau)$, all functions of τ in (8) become functions of the norm r on \mathcal{L} , thus giving rise to a metric defined on \mathcal{L} . Furthermore, there exists a positive value r_0 for which $\tau(r_0) = 0$, and the disc bundle S is characterized by the inequality $r \leq r_0$.

A version of the main theorem of [DM2], Theorem 16.3 there, now holds for M with essentially the same proof, i.e.

Theorem 2. *Let M be a compact manifold with a nonempty boundary, with a Kähler metric g and a special Kähler-Ricci potential τ on its interior, both extending smoothly to the boundary. Suppose τ has the value zero on the boundary. Then the triple (M, g, τ) is biholomorphically isometric to the disc bundle (or disc) $(S, \bar{g}, \tau(r))$ described above, and the orbits of the Killing vector field $J\nabla\tau$ are closed.*

Note that the assumption on the boundary value of τ excludes the possibility of S being an annulus bundle, and guarantees that M has a critical manifold, the preimage of the nonzero extremum τ_0 of τ . Also, the normal exponential map which is a building block for the inverse of the biholomorphic isometry gives a diffeomorphism $S \setminus \{r = r_0\} \rightarrow M \setminus \partial M$ (this is [DM2, Lemma 13.2], adjusted to the case of a manifold with boundary).

The last part of the conclusion of the theorem, which can be considered a consequence of the fact that the orbits of $J\nabla\tau$ coincide under the above biholomorphic isometry with the S^1 orbits of the natural circle action on L , can be seen directly as follows. At a critical point of a special Kähler-Ricci potential τ , the Hessian

$\nabla d\tau$ has exactly one nonzero eigenvalue (see [DM2, Proposition 7.3]). By [DM2, Lemma 10.2], the vector field $J\nabla\tau$, being the image under the differential of the exponential map of a linear vector field with a periodic flow, has itself a periodic flow in a neighborhood of the critical point. By the unique continuation property for isometries this periodicity is in fact global, as two isometries in the flow that agree on a nonempty open set actually agree globally. However, comparing isometries of a flow, and, more specifically, verifying the existence of a circle action produced by a flow [DM2, Corollary 10.3], may be performed if the vector field is complete. In [DM2] completeness of the metric, which implies completeness of the vector field, was assumed. However only the latter is necessary, and in the setting at hand, of a manifold with boundary, it occurs automatically. In fact, as g and τ are smooth everywhere, $\nabla\tau$ and the Killing field $J\nabla\tau$ are well-defined on the boundary, and since the latter is orthogonal to the former, and $\nabla\tau$ is normal to the boundary, $J\nabla\tau$ is in fact tangent to the boundary at boundary points, and hence necessarily complete. (Note as an aside that a chain of reasoning exists which, starting with the fact that $J\nabla\tau$ is Killing, concludes with smoothness of τ . Thus in the statement of Theorem 2, it is enough to assume continuity of τ at the boundary.)

Remark 3.1. If we now add to the above assumptions the requirement on (M, g) that it is a (Kählerian) conformal compactification of an *Einstein* metric on the interior and M is of dimension four, the metric \bar{g} (and hence indirectly g) can be specified by a local formula for Q [DM1, §19-21]. In fact, $Q(\tau)$ is a member of one of the following three families of rational functions, whose formulas are given below with K, α, β, A, B and C denoting constants (the precise choice of which ensures that $Q(\tau)$ satisfies the boundary and positivity conditions (5)):

- a) $Q = -K\tau^2 + (\alpha\tau^3 - \beta/2)/3$ if $\phi = 0$.
- b) $Q = -K\tau/2 + \alpha\tau^3 - \beta/3$ if $\phi \neq 0$ and $c = 0$.
- c) $Q = (\tau/c - 1)(AE(\tau/c) + BF(\tau/c) + C)$ if $\phi \neq 0$ and $c \neq 0$, where
 $E(x) = x^2 - 1, \quad F(x) = (x - 2)x^3/(x - 1)^2$.

These formulas are given for completeness, and will not be used again.

3.2. The dual Kähler metric. As in §3.1, Let (M, g, τ) be a triple consisting of a compact manifold M with a nonempty boundary, having a defining function τ , and a Kähler metric g on the interior of M for which τ is a special Kähler-Ricci potential. Suppose also that the isometric metric \bar{g} of the last section is either of type ii] or type iii] in (8). Such a metric was called *nontrivial SKR* in [M]. In this case the constant c is defined (see paragraph between (5) and (6)) and in fact

dual

$$(9) \quad \hat{g} = g/(\tau - c)^2 \text{ is Kähler for an oppositely oriented complex structure}$$

while $\widehat{t} = 1/(\tau - c)$ is a special Kähler-Ricci potential for \widehat{g} (see [M, Proposition 5.1] or [DM3, Remark 28.4]). Our main use of this fact is in the formula

$$(10) \quad |W_-|^{\text{hat}} = (\tau - c)^{-4} |\widehat{W}_+|_{\widehat{g}}^2$$

which follows since W_- is, as a $(3, 1)$ tensor, a conformal invariant (and using the conformal factor $(\tau - c)^2$ relating g with \widehat{g}).

4. MOMENT MAP FORMULAS FOR THE η INVARIANT

4.1. The case of a special Kähler-Ricci potential. In the following theorem, the meaning of the notations N , h , c , e , τ_0 , r_0 and p is that given in §3.1, and Theorem 2 is implicitly used in its proof. Thus we associate with the Kähler metric g data used in the construction of its isometric metric \bar{g} . Additionally, we continue to denote the signature of a manifold by $\sigma(M)$, while the volume of N with respect to h will be denoted $Vol_h(N)$.

Theorem 3. *Let M^4 be a compact four-manifold having a nonempty boundary with a complete Einstein metric g_E on its interior, admitting a Kählerian conformal compactification (g, τ) satisfying (4) on its interior for a nontrivial SKR metric. Then the eta invariant of the boundary metric $\gamma = g|_{\partial M}$ is given by*

$$(11) \quad \eta_\gamma(\partial M) = \frac{1}{288\pi^2} Vol_h(N) \int_0^{\tau_0} [(b^2(t - c)^{-6} - a^2) t^2] \cdot [l + pt] dt - \sigma(M)$$

for some nonzero constants a and b , with l standing for the constant $2|c|$ if τ has a complex codimension one critical manifold, and $2|c|/(|e|r_0^2) = 2|\tau_0|/(|e|r_0^2)$ (see (6)) if the τ -critical manifold consists of one point.

Remark 4.1. The integral in (11), is of course, elementary and equals

$$\left[-\frac{t^2 a^2 p}{2} - a^2 l t - \frac{b^2 p}{4(t - c)^4} - \frac{b^2(l + cp)}{5(t - c)^5} \right] \Big|_0^{\tau_0}.$$

Proof. By Theorem 1, formula (1) gives an expression for the eta invariant in terms of the signature, the Weyl tensors W_+ , W_- and the volume form of g . Using the dual metric \widehat{g} , one can replace $|W_-|^2$ by expression (10) involving $|\widehat{W}_+|_{\widehat{g}}^2$. Since both g and \widehat{g} are Kähler metrics, the self-dual part of their Weyl tensors can be expressed in terms of the scalar curvatures s and \widehat{s} (as in (3), cf. [D]). We thus have

$$(12) \quad \begin{aligned} \eta_\gamma(\partial M) &= \frac{1}{12\pi^2} \int_M [(\tau - c)^{-4} |\widehat{W}_+|_{\widehat{g}}^2 - |W_+|^2] \text{vol}_g - \sigma(M) \\ &= \frac{1}{288\pi^2} \int_M [(\tau - c)^{-4} \widehat{s}^2 - s^2] \text{vol}_g - \sigma(M) \end{aligned}$$

Next, in dimension four, the scalar curvature of a Kähler metric which is conformal to an Einstein metric is a constant nonzero multiple of the square root of the conformal

factor (cf. [DM1, Proposition 6.5]). That square root is τ for g , and $\tau/(\tau - c)$ for $\widehat{g} = g/(\tau - c)^2$. Thus, writing $s = a\tau$, $\widehat{s} = b\tau/(\tau - c)$ for nonzero constants a , b and ω for the Kähler form of g we get

$$(13) \quad \eta_\gamma(\partial M) = \frac{1}{288\pi^2} \int_M [(b^2(\tau - c)^{-6} - a^2) \tau^2] \omega^2/2 - \sigma(M).$$

We now apply the Duistermaat-Heckman Theorem [DH] to the integral in (13). Namely, the Killing potential τ is a moment map for the symplectic form ω associated with g , with respect to the circle action guaranteed by Theorem 2. The value $\tau = 0$ is regular for this moment map, because $|\nabla\tau|_{\tau=0} \neq 0$, by (5). The theorem states that the Duistermaat-Heckman measure, i.e. the moment map push-forward of the measure associated with the symplectic form on M , is the function $[\omega_0 + t\Omega][N]$, with ω_0 denoting the induced symplectic form on the associated reduced space $N = \tau^{-1}(0)/S^1$ and Ω denoting the curvature of L (See [G, Excercise 2.18]). We have

$$(14) \quad \begin{aligned} \eta_\gamma(\partial M) &= \frac{1}{288\pi^2} \int_0^{\tau_0} [(b^2(t - c)^{-6} - a^2) t^2] ([\omega_0 + t\Omega][N]) dt - \sigma(M) \\ &= \frac{1}{288\pi^2} \int_0^{\tau_0} [(b^2(t - c)^{-6} - a^2) t^2] ([\omega_0 + t p \omega^h][N]) dt - \sigma(M) \\ &= \frac{1}{288\pi^2} \text{Vol}_h(N) \int_0^{\tau_0} [(b^2(t - c)^{-6} - a^2) t^2] [l + p t] dt - \sigma(M). \end{aligned}$$

The second line follows from (7). The third follows from cases ii] and iii] of (8), as ω_0 is the solution of the equation $\pi^*\omega_0 = \iota^*\omega$, valid on ∂M with ι being the inclusion $\partial M \hookrightarrow M$ and π the projection of ∂M to the reduced space $\partial M/S^1$. \square

Remark 4.2. The value of the signature $\sigma(M)$ in the above formula is in fact 0 or -1 in the case where M is homeomorphic to a disk bundle S over a Riemann surface N . Contractibility of the disk implies that $H_2(S) = H_2(N) = \mathbf{Z}$, so that the intersection form is a number, namely the Euler number of the bundle S . The group $H_2(S)$ is generated by a section, and the intersection matrix is determined by its nonpositive self-intersection number.

4.2. The ambitoric case. Ambihhermitian structures were defined in [ACG]. These consist, on a 4-manifold M , of a conformal class c , with two c -orthogonal complex structures J, \widehat{J} inducing opposite orientations. A metric in such a conformal class is called ambihhermitian. Suppose such a metric g_E is Einstein, and degeneracy of the Weyl tensor components W_+ and W_- holds, in the sense that as operators on selfdual (respectfully antiselfdual) 2-forms, at least two of their three eigenvalues coincide. Then in a neighborhood of each point the structure is ambi-Kähler, i.e. there exist two metrics g, \widehat{g} in the conformal class c such that $(g, J), (\widehat{g}, \widehat{J})$ are Kähler [ACG, Proposition 1]. The three metrics satisfy $g_E = s^{-2}g = \widehat{s}^{-2}\widehat{g}$ on an open and dense set (away from the zero sets of s or \widehat{s}), where s (\widehat{s}) is the scalar curvature of g (\widehat{g}).

Furthermore, $J\nabla s$ and $\widehat{J}\widehat{\nabla}\widehat{s}$ are Killing vector fields with respect to both g and \widehat{g} [ACG, Proposition 7], so that $\widehat{J}\widehat{\nabla}\widehat{s} = J\nabla f$ and $J\nabla s = \widehat{J}\widehat{\nabla}\widehat{f}$ for two functions f, \widehat{f} . Generically, these vector fields span a 2-dimensional space of Poisson-commuting Killing vector fields, and the whole structure is called ambitoric. We will also assume that (s, f) and $(\widehat{f}, \widehat{s})$ are each moment maps for a toric group action (in the case of a manifold without boundary, this follows from the assumption of having an ambitoric structure). We now show that in this case, a formula also exists for the eta invariant of the boundary.

Theorem 4. *Let M^4 be a compact four-manifold having a nonempty boundary with a complete Einstein metric g_E whose conformal class admits an ambitoric structure consisting of the metrics g, \widehat{g} , with all metrics defined on the interior. If the vector fields $J\nabla s$ and $\widehat{J}\widehat{\nabla}\widehat{s}$ span an ambitoric structure and each gives rise to a toric structure as described above, then the eta invariant of the boundary metric $\gamma = g|_{\partial M}$ is given by*

$$\eta_\gamma(\partial M) = \frac{1}{288\pi^2} \int_{\widehat{P}} \widehat{y}^2 v_{euc} - \frac{1}{288\pi^2} \int_P x^2 v_{euc} - \sigma(M),$$

where P, \widehat{P} are the images of the moment maps $(s, f), (\widehat{f}, \widehat{s})$ respectively. Here x, \widehat{y} are the functions on these images of the moment maps whose pullbacks are s and \widehat{s} , respectively, and v_{euc} represents integration with respect to Lebesgue measure.

Proof. Suppose f is the function such that $g = f^2 \widehat{g}$. We begin by noticing that the relation generalizing (10) is $|W_-|^2 = f^{-4} |\widehat{W}_+|_{\widehat{g}}^2$, so that by (1)

$$\eta_\gamma(\partial M) = \frac{1}{12\pi^2} \int_M \left[f^{-4} |\widehat{W}_+|_{\widehat{g}}^2 - |W_+|^2 \right] \text{vol}_g - \sigma(M).$$

Next, $\text{vol}_g = f^4 \text{vol}_{\widehat{g}}$, so that in the first line below, f cancels to give:

$$\begin{aligned} \eta_\gamma(\partial M) &= \frac{1}{12\pi^2} \int_M |\widehat{W}_+|_{\widehat{g}}^2 \text{vol}_{\widehat{g}} - \frac{1}{12\pi^2} \int_M |W_+|^2 \text{vol}_g - \sigma(M) \\ &= \frac{1}{288\pi^2} \int_M \widehat{s}^2 \text{vol}_{\widehat{g}} - \frac{1}{288\pi^2} \int_M s^2 \text{vol}_g - \sigma(M) \\ &= \frac{1}{288\pi^2} \int_{\widehat{P}} \widehat{y}^2 v_{euc} - \frac{1}{288\pi^2} \int_P x^2 v_{euc} - \sigma(M). \end{aligned}$$

The last line follows from the fact that the Duistermaat-Heckman measure in the toric case equals Lebesgue measure on the image of the moment map. \square

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