Scalar curvature and holomorphy potentials

Gideon Maschler

Department of Mathematics and Computer Science, Clark University, Worcester, Massachusetts 01610, U.S.A.

Abstract

A holomorphy potential is a complex valued function whose complex gradient, with respect to some Kähler metric, is a holomorphic vector field. Given k holomorphic vector fields on a compact complex manifold, form, for a given Kähler metric, a product of the following type: a function of the scalar curvature multiplied by functions of the holomorphy potentials of each of the vector fields. It is shown that the stipulation that such a product be itself a holomorphy potential for yet another vector field singles out critical metrics for a particular functional. This may be regarded as a generalization of the extremal metric variation of Calabi, where $k = 0$ and the functional is the square of the L^2-norm of the scalar curvature. The existence question for such metrics is examined in a number of special cases. Examples are constructed in the case of certain multifactored product manifolds. For the SKR metrics investigated by Derdzinski and Maschler and residing in the complex projective space, it is shown that only one type of nontrivial criticality holds in dimension three and above.

Key words: holomorphy potential, Killing potential, extremal Kähler metric
2010 MSC: 53C55, 53C25, 58E11

1. Introduction

Extremal Kähler metrics were defined and studied by Calabi [2]. In his framework they are obtained from the variation of a curvature functional, over Kähler metrics (with Kähler forms) in a fixed deRham cohomology class of a compact complex manifold. The resulting Euler-Lagrange equation is equivalent to the requirement that the scalar curvature of the critical metric is a *holomorphy potential*. In other words, the complex gradient of the scalar curvature of the critical metric is a holomorphic vector field.

In this paper we will be concerned with a more general requirement on the scalar curvature. To put it in context, consider first the stipulation that

$$\text{the scalar curvature functionally depends on a collection of holomorphy potentials.} \tag{1.1}$$

This condition is satisfied by a subset of the class of Kähler manifolds with rigid torus action, which includes metrics with Hamiltonian two forms, toric metrics, as well as other classes [1]. They are defined by requiring the values of the metric on vector fields tangent to a given isometric torus action be functions of the Killing potentials of these vector fields. Such manifolds are locally classified to be principal torus bundles over a Kähler base, and condition (1.1) is satisfied exactly when the Kähler base metric has constant scalar curvature.

Email address: gmaschler@clarku.edu (Gideon Maschler)

The condition we investigate in this article is closely related, but also significantly more stringent. Namely, we require that

$$\varphi_{k+1} := p(s)h_1(\varphi_1) \cdot \ldots \cdot h_k(\varphi_k) \text{ is a holomorphy potential,} \tag{1.2}$$

where s is the scalar curvature, each φ_i, $i = 1 \ldots k$ is a holomorphy potential while $p : \mathbb{R} \to \mathbb{R}$, $h_i : \mathbb{C} \to \mathbb{C}$, $i = 1 \ldots k$ are arbitrary smooth functions.

One reason to concentrate on condition (1.2) is that it appears as the Euler-Lagrange requirement for a functional variation which naturally generalizes the extremal metric case. In fact, the functional has the form

$$S = S_{f,h_1 \ldots h_k} := \int_{M^m} f(s)h_1(\varphi_1) \cdot \ldots \cdot h_k(\varphi_k)\, \omega^{\wedge m},$$

and its variation gives (1.2) for $p = f'$. The special case of extremal metrics corresponds to the choice $k = 0$ and $f(x) = x^2$, i.e. when the functional is the square of the L^2-norm of the scalar curvature. Note that the variation takes place in a given Kähler class while fixing a background metric and k holomorphic vector fields (see §2.2). The functions f and h_i, $i = 1 \ldots k$ are treated as parameters that must be specified in order to determine the functional.

Returning to condition (1.2), one major distinction is to be made. The vector field produced by the variation with holomorphy potential φ_{k+1} may be linearly dependent on the k vector fields fixed in defining the variation. Alternatively, it may be "genuinely new", i.e. this vector field, together with the fixed ones, form a linearly independent set. We explore both cases, focusing mainly on the latter, which is more involved.

In §2.3 we examine for $k = 1$ the case of linear dependence, where the constructed vector field is proportional to the initial fixed one. This case is roughly equivalent to the requirement that the critical metric satisfying $s = H(\varphi)$, with $\varphi = \varphi_1$ and some smooth $H : \mathbb{C} \to \mathbb{R}$. A well understood class of functions fulfilling the latter condition is provided by the class of SKR metrics [3] (see §3.1 for their definition). These metrics where initially defined due to their role in the classification of Kähler metrics conformal to an Einstein metric (an example for the latter is the metric constructed by the physicist D. Page [8]).

In §3.1 we construct, for any k, a critical metric with a vector field which forms a linearly independent set with the fixed ones. This metric is a product of certain SKR metrics. In §3.2 we prove Theorem 3.1, whose first part consists of a characterization of SKR metrics among a family of metrics given on a complex vector space, and an explicit determination of the associated holomorphy potentials of linear vector fields. In the proof of the second part of this theorem we examine whether such metrics, aside from their role as critical metrics in the linearly dependent case, may also serve as critical metrics (for $k = 1$), in the case where the constructed vector field and the fixed one are linearly independent. We find a dimensional obstruction for this to hold. For some of these SKR metrics the result extends from the vector space to the complex projective space.

2. A variational characterization

2.1. Vector fields - real and complex viewpoints

The material presented in this subsection is roughly equivalent to that in [5, Theorem 4.4 and corollaries]. However, our discussion proceeds mostly via real, as opposed to complex, geometric terms. Moreover, our aim is to relate the real and complex viewpoints concretely, and in this we go one small step further than the above reference.

Let (M, g) be a compact Kähler manifold with associated almost complex structure $J : TM \to TM$. Thus $J^2 = -1$, and J is skew-adjoint and parallel with respect to the Riemannian metric g. The Kähler form of g is given by $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$, and the $\mathbb{C}$-valued sesquilinear form whose real part is ω is $\omega^{\mathbb{C}} = (\omega - i\omega(J\cdot, \cdot))/2 = (\omega + ig)/2$. Finally, the induced action of J on 1-forms is $J^*\xi(\cdot) = \xi(J\cdot)$.

We consider further the action of J on 1-forms. Let Ω^k denote the space of k-forms on M, $k = 1 \ldots \dim M$. We write $d\Omega^k$, $d^*\Omega^k$ for the spaces of exact $(k+1)$-forms and coexact $(k-1)$-forms, respectively, and $\mathcal{H}$ for the space of harmonic 1-forms. We have

$$J^*(d\Omega^0) \text{ is } L^2\text{-orthogonal to } d\Omega^0 \text{ and to } \mathcal{H}. \tag{2.1}$$

In fact, by the Hodge decomposition the space $d\Omega^0 \oplus \mathcal{H}$ is exactly the space of closed 1-forms (since for a closed 1-form, the term of the Hodge decomposition in $d^*\Omega^2$ is both closed and coclosed, i.e. harmonic and therefore zero). Now for a real valued function α and a closed 1-form ν, we have $\int \langle J^* d\alpha, \nu \rangle = \int J_p^q \alpha_{,q} \nu^p = -\int \alpha\, \omega_{pq} \nu^{p,q} = 0$, as $2\omega_{pq}\nu^{p,q} = \omega_{pq}(d\nu)^{qp}$. This completes the verification of (2.1), which implies, again by the Hodge decomposition,

$$J^*(d\Omega^0) \subseteq d^*\Omega^2. \tag{2.2}$$

Suppose X is a complex vector field of type $(1,0)$, so that $X = (u - iJu)/2$ for a vector field u, and let $\varphi = \alpha + i\beta$ be a complex valued function. Then

$$\imath_X \omega^{\mathbb{C}} = \bar{\partial}\varphi \quad \Longleftrightarrow \quad \imath_u g = d\beta + J^* d\alpha, \tag{2.3}$$

where $\imath$ denotes interior multiplication. This can be seen by substituting for X, as well as for $\bar{\partial}\varphi = (d\varphi + iJ^* d\varphi)/2$ with $d\varphi = d\alpha + i d\beta$, and separating real and imaginary parts.

Assume X is holomorphic, that is, has holomorphic components in a some complex coordinate system around each point. Then by [5, Corollary 4.5], it satisfies the left hand side of (2.3) for some function φ if it has a nonempty zero set. Hence the right hand side also holds, and we presently examine α and β in a special case, namely that of a Killing vector field. The function φ is called a *holomorphy potential* for X. Note that for a given holomorphic vector field X, the holomorphy potential varies with the metric, and this will be examined in the next subsection.

Suppose τ is a *Killing potential*, i.e a smooth real valued function for which $J\nabla\tau$ is a Killing vector field. We set $v = \nabla\tau$ and $u = Jv$. Thus u is an infinitesimal generator for isometries, that is, $\mathcal{L}_u g = 0$, with $\mathcal{L}$ denoting the Lie derivative. By [3, Lemma 5.2], v is holomorphic, that is, $\mathcal{L}_v J = 0$, which is equivalent to $X = (u - iJu)/2 = i(v - iJv)/2$ being holomorphic in the sense of the previous paragraph. Also v, u (and hence X) each have a nonempty zero set consisting of the critical points of τ. Following the discussion in the previous paragraph, X, and hence u, satisfy the conditions in (2.3) for some function $\varphi = \alpha + i\beta$. We wish to describe how τ is related to φ.

As is well known, u is Killing if and only if ∇u is skew-adjoint. Thus u is divergence-free, and taking the divergence on the right hand side equation in (2.3), gives

$$0 = \Delta_d \beta + d^* J^*(d\alpha),$$

with Δ_d denoting the Laplacian. However, by (2.2) $d^* J^*(d\alpha) = 0$, implying that

$$\beta = \text{constant}.$$

Now set $\xi = \imath_u g$. By taking d of the right hand side of (2.3) we see that $d\xi = dJ^* d\alpha$. Combining this with $d\xi(\cdot,\cdot) = 2\nabla d\tau(J\cdot,\cdot)$ (see [3, Lemma 5.5]) gives

$$2\nabla d\tau(J\cdot,\cdot) = d[J^*(d\alpha)](\cdot,\cdot) = -2\cdot \operatorname{skew}[\nabla d\alpha(J\cdot,\cdot)], \tag{2.4}$$

where "skew" denotes the skewsymmetric component. Note that $\nabla d\tau$ is Hermitian by [3, Lemma 5.2 (iii)] so the left hand side of (2.4) is already skewsymmetric. In particular, to represent X via a holomorphy potential φ one can always take

$$\varphi = \alpha = -\tau.$$

In other words, we can choose the holomorphy potential of the holomorphic vector field corresponding to a Killing vector field with a nonempty zero set to be, up to sign, the Killing potential.

Finally, note that a converse also holds. If $X = (u - iJu)/2$ is holomorphic with a *real* holomorphy potential $-\tau$, then u is Killing with Killing potential τ. In fact, for $v = -Ju$ we have that $-iX = (v - iJv)/2$ is also holomorphic, and this is equivalent to v being holomorphic (in the sense that $\mathcal{L}_v J = 0$). Also, $u = Jv$ and $\imath_u g = J^* d\alpha = -J^* d\tau$ (which is the right hand side of (2.3) in this case) imply, by composing the latter relation with J, that $v = \nabla\tau$. Hence v is a holomorphic gradient, which, by [3, Lemma 5.2] implies that τ is a Killing potential for the Killing field u.

Due to the above, in the cases below where both notations are used, φ and τ will be nearly interchangeable.

2.2. Functionals

In this section we review a lemma of Calabi and then construct a rather direct generalization to his result on the variation that leads to, and in fact defines, extremal Kähler metrics. We assume throughout that (M, g) is a compact Kähler manifold of complex dimension m. In the next lemma we employ complex coordinates z^a, with commas denoting complex covariant differentiation.

Lemma 2.1 (Calabi)**.** *If a complex valued function φ satisfies the equation $\varphi^{,ab}{}_{ab} = 0$, then $\varphi_{,\bar{a}\bar{b}} = 0$. In other words, φ is a holomorphy potential (as $\varphi^{,a}{}_{\bar{b}} = 0$).*

Proof. Write

$$0 \leq g^{a\bar{c}} g^{b\bar{d}} \varphi_{,\bar{c}\bar{d}} \bar{\varphi}_{,ab} = (\varphi^{,ab} \bar{\varphi}_{,a} - \varphi^{,ab}{}_{a} \bar{\varphi})_{,b} + \varphi^{,ab}{}_{ab} \bar{\varphi} := v^b_{,b} + \varphi^{,ab}{}_{ab} \bar{\varphi},$$

which follows since $\varphi^{,ab}{}_{b} = \varphi^{,ba}{}_{b}$. Under the condition of the lemma, the first term on the right hand side is the divergence of the $(0, 1)$ part of the form corresponding to v:

$$v^b_{,b} = - * d * (v_{\bar{b}} dz^{\bar{b}}),$$

which vanishes upon integration, so that the integral of the left hand side also vanishes. That integral is the square of the L^2 norm of the tensor $\varphi_{,\bar{a}\bar{b}}$, hence the result follows. ☐

Proposition 2.2. *On a compact complex manifold M with $\dim_{\mathbb{C}} M = m$, fix objects $\left(\widehat{g}, \{X_i\}_{i=1}^k\right)$, consisting of a (background) Kähler metric $\widehat{g}$ with Kähler form $\widehat{\omega}$, and for $i = 1, \ldots k$, a holomorphic vector field (with zeros) X_i admitting a $\widehat{g}$-holomorphy potential $\widehat{\varphi}_i$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $h_i : \mathbb{C} \to \mathbb{C}$ are smooth functions $(i = 1 \ldots k)$. Consider the functional*

$$S = S_{f,h_1,\ldots h_k} := \int_M f(s) \prod_{i=1}^k h_i(\varphi_i)\, \omega^{\wedge m},$$

defined over the space of Kähler metrics (with Kähler forms) in the fixed Kähler class $[\widehat{\omega}]$. Here, a given such metric g has Kähler form ω and scalar curvature s. Also, for i=1…k, each X_i has g-holomorphy potential φ_i normalized via the condition $\int_M \varphi_i\, \omega^{\wedge m} = \int_M [\widehat{\varphi}_i + X_i u]\, \omega^{\wedge m}$, where u is the smooth purely imaginary function given up to a constant by $\omega = \widehat{\omega} + \partial\bar{\partial} u$. A metric g satisfies the Euler-Lagrange equation for S if and only if the function

$$\varphi_{k+1} = f'(s) \prod_{i=1}^k h_i(\varphi_i) \tag{2.5}$$

is also a g-holomorphy potential (and, of course, is a g-Killing potential if it is real).

Here the prime denotes differentiation while $f(s)$ stands for the composition $f \circ s$, and similarly for $h_i(\varphi_i)$. Unless all h_i, $i = 1, \dots k$ are chosen to be real valued, Theorem 2.2 describes a variation of a complex valued functional. However, our main interest will be the case where the critical equation (2.5) gives rise not just to a holomorphy potential, but to a Killing potential with respect to the critical metric. Finally, the meaning of the normalization condition for φ_i will be clarified in §2.3.1.

Proof. We first recall Calabi's basic variation scheme. As the variation takes place in a fixed Kähler class, the Kähler forms in a one parameter family g_t, may be considered to vary according to $\omega_t = \widehat{\omega} + \partial\bar{\partial} u_t$, for purely imaginary functions u_t on the product of M with an open interval. The t-derivative at zero of u_t will be denoted υ, and similar derivatives involving other geometric quantities will be denoted using the symbol δ. One has $\delta g_{a\bar{b}} = \upsilon_{,a\bar{b}}$. Next, as variations of determinants involve traces, $\delta(\det g_{a\bar{b}}) = \Delta\upsilon \det g_{a\bar{b}}$, or $\delta(\omega^{\wedge m}) = \Delta\upsilon(\omega^{\wedge m})$. Here Δ is the $\bar{\partial}$-Laplacian, $\Delta\upsilon = -g^{a\bar{b}}\nabla_a\nabla_{\bar{b}}\upsilon$. Consequently the Ricci tensors $(r_t)_{a\bar{b}} = -\left(\log\det(g_t)\right)_{,a\bar{b}}$ admit the variation $\delta r_{a\bar{b}} = (\Delta\upsilon)_{,a\bar{b}}$.

Next, the variation of the scalar curvatures $s_t = \langle r_t, g_t\rangle_t$ for the one parameter family g_t depends on r_t, g_t and the inner products $\langle\cdot,\cdot\rangle_t$. For this notice that from $g \cdot g^{-1} = Id$ it follows that $\delta(g^{a\bar{c}})g_{b\bar{c}} = -g^{a\bar{c}}\delta(g_{b\bar{c}})$, and so, with R standing for the full curvature tensor,

$$\begin{aligned}
\delta s &= g^{a\bar{b}}\delta r_{a\bar{b}} + \delta(g^{c\bar{b}}g^{a\bar{d}}g_{c\bar{d}})r_{a\bar{b}} = -\Delta^2\upsilon + (1-2)g^{c\bar{b}}g^{a\bar{d}}\delta(g_{c\bar{d}})r_{a\bar{b}} \\
&= -\Delta^2\upsilon - \upsilon^{,a\bar{b}}r_{a\bar{b}} = -\upsilon_{,a}{}^{a}{}_{b}{}^{b} - \upsilon_{,a}{}^{b}r_{b}{}^{a} \\
&= (-\upsilon_{,ab}{}^{a} + \upsilon_{,d}R^{d}{}_{ab}{}^{a})^{,b} - \upsilon_{,a}{}^{b}r_{b}{}^{a} = -\upsilon_{,ab}{}^{ab} + \upsilon_{,d}{}^{b}r^{d}{}_{b} + \upsilon_{,d}\, r^{d}{}_{b}{}^{,b} - \upsilon_{,a}{}^{b}r_{b}{}^{a} \\
&= -\upsilon_{,ab}{}^{ab} + \upsilon_{,d}s^{,d}\ ,
\end{aligned}$$

where the symmetry of both the Hessian of υ and r was used in the second and third lines, the Ricci-Weitzenböck formula in the third line, and the contracted Bianchi identity in the last line.

We proceed to the variation of the holomorphy potentials $(\varphi_i)_t$ of g_t, for a fixed holomorphic vector field X_i, with its $\widehat{g}$-holomorphy potential denoted by $\widehat{\varphi}_i$ as in the statement of the proposition. For Kähler forms $\omega_t = \widehat{\omega} + \partial\bar{\partial}u_t$, we have $\imath_{X_i}\omega_t = \imath_{X_i}\omega + \imath_{X_i}(\partial\bar{\partial}u_t) = \bar{\partial}\widehat{\varphi}_i - \imath_{X_i}(\bar{\partial}\partial u_t) = \bar{\partial}\widehat{\varphi}_i + \bar{\partial}(\imath_{X_i}\partial u_t) = \bar{\partial}(\widehat{\varphi}_i + X_i u_t)$, where we have used the fact that $\imath_{X_i}$ anti-commutes with $\bar{\partial}$ when X_i is holomorphic. Hence $(\varphi_i)_t = \widehat{\varphi}_i + X_i u_t + (c_i)_t$ for some t-dependent constant $(c_i)_t$. But the normalization of the holomorphy potentials implies $\int_M[\widehat{\varphi}_i + X_i u_t + (c_i)_t]\,\omega_t^{\wedge m} = \int_M(\varphi_i)_t\omega_t^{\wedge m} = \int_M[\widehat{\varphi}_i + X_i u_t]\,\omega_t^{\wedge m}$, so that $\int_M(c_i)_t\,\omega_t^m = 0$. Hence, with this normalization, the constants $(c_i)_t$ all vanish and $(\varphi_i)_t = \widehat{\varphi}_i + X_i u_t$. Therefore the variation, i.e. the derivative at $t = 0$, is

$$\delta\varphi_i = X_i\upsilon = \langle\partial\varphi_i, \partial\upsilon\rangle = \langle\partial\, Re\,\varphi_i, \partial\upsilon\rangle + i\langle\partial\, Im\,\varphi_i, \partial\upsilon\rangle,$$

and the last two terms are, of course, $\delta Re\,\varphi_i$ and $i\delta Im\,\varphi_i$, respectively.

Therefore, denoting by $(h_i)_x$, $(h_i)_y$ the partial derivatives of h_i with respect to $x = Re\,z$ and $y = Im\,z$, we have

$$\delta\Big(f(s)\prod_{\ell=1}^{k} h_\ell(\varphi_\ell)\omega^{\wedge m}\Big) = \Big(f'(s)\left(-\upsilon_{,ab}{}^{ab} + \upsilon_{,d}s^{,d}\right)\prod_{\ell=1}^{k} h_\ell(\varphi_\ell)$$

$$+ \quad f(s)\sum_{j=1}^{k}\Big(\Big[(h_j)_x(\varphi_j)\delta(Re\,\varphi_j) + i\,(h_j)_y(\varphi_j)\delta(Im\,\varphi_j)\Big]\prod_{\substack{\ell=1\\ \ell\neq j}}^{k} h_\ell(\varphi_\ell)\Big) + f(s)\prod_{\ell=1}^{k} h_\ell(\varphi_\ell)\Delta\upsilon\Big)\omega^{\wedge m}$$

$$= \quad \Big(-f'(s)\prod_{\ell=1}^{k} h_\ell(\varphi_\ell)\upsilon_{,ab}{}^{ab}$$

$$+ \quad \Big\langle f'(s)\prod_{\ell=1}^{k} h_\ell(\varphi_\ell)\partial s + f(s)\sum_{j=1}^{k}\Big(\Big(\prod_{\substack{\ell=1\\ \ell\neq j}}^{k} h_\ell(\varphi_\ell)\Big)\Big[(h_j)_x(\varphi_j)\partial\left(Re\,\varphi_j\right) + i\,(h_j)_y(\varphi_j)\partial\left(Im\,\varphi_j\right)\Big]\Big), \partial\upsilon\Big\rangle$$

$$+ f(s)\prod_{\ell=1}^{k} h_\ell(\varphi_\ell)\Delta\upsilon\Big)\omega^{\wedge m} = \Big(-f'(s)\prod_{\ell=1}^{k} h_\ell(\varphi_\ell)\upsilon_{,ab}{}^{ab} + \operatorname{div}(f(s)\prod_{\ell=1}^{k} h_\ell(\varphi_\ell)\partial\upsilon)\Big)\omega^{\wedge m},$$

using only the Leibniz rule, with div denoting the divergence operator. Since the second summand is indeed a divergence, its integral vanishes. Therefore,

$$\delta S = -\int_{M^m} f'(s)\cdot\prod_{i=1}^{k} h_i(\varphi_i)\cdot\upsilon_{,ab}{}^{ab}\,\omega^{\wedge m} = -\int_{M^m}\left[(f'(s))\cdot\prod_{i=1}^{k} h_i(\varphi_i)\right]^{,ba}_{\quad ba}\cdot\upsilon\,\omega^{\wedge m},$$

as one sees integrating by parts four times. But equating this to zero is a requirement that must hold for every υ, hence one arrives at

$$\left[f'(s)\cdot\prod_{i=1}^{k} h_i(\varphi_i)\right]^{,ba}_{\quad ba} = 0.$$

The result now follows from Lemma 2.1

□

2.3. Remarks on special cases

The following is a description of some examples of metrics with a holomorphy potential given by (2.5).

2.3.1. Tautological examples leading to invariants

In some cases Equation (2.5) gives the holomorphy potential of the zero vector field. Obviously this can be achieved by choosing f or one of the functions h_i to be the zero function. Another more interesting possibility is to have f be a nonzero constant function. In that case, the integrand of S is a product of functions of the φ_i, while Equation (2.5) yields again zero as a holomorphy potential. As this is a holomorphy potential for *any* metric, any metric is critical, hence the functional S must be constant, i.e. an invariant associated with the Kähler class. This well-known fact can be understood in the context of holomorphic equivariant cohomology. Namely, let $k = 1$ and write $X := X_1$, $\widehat{\varphi} := \widehat{\varphi}_1$, $\varphi := \varphi_1$ and $h := h_1$. Denote by $\bar{\partial}$ the Dolbeault operator, and $\imath_X$, as before, interior multiplication by X. Then the operator $\bar{\partial} - \imath_X$ acts on equivariant forms, i.e forms in $A_l := \oplus_{q-p=l} A^{p,q}$, with $A^{p,q}$ denoting (p,q)-forms (cf. [6]). Consider for simplicity the case where h is the identity function, so that the functional is a multiple of $\int_M \varphi\omega^{\wedge m}$. Then the otherwise mysterious normalization condition in Proposition 2.2 can be employed as follows: $(m+1)\int_M \varphi\,\omega^{\wedge m} = (m+1)\int_M [\widehat{\varphi}+Xu]\,\omega^{\wedge m} = (m+1)\int_M [\widehat{\varphi}+Xu]\,\left(\widehat{\omega}+\partial\bar\partial u\right)^{\wedge m} = \int_M [\widehat{\omega}+\partial\bar\partial u+\widehat{\varphi}+Xu]^{\wedge(m+1)} = \int_M [\widehat{\omega}+\widehat{\varphi}+\partial\bar\partial u+Xu]^{\wedge(m+1)} = \int_M [\widehat{\omega}+\widehat{\varphi}-(\bar\partial-\imath_X)(\partial u)]^{\wedge(m+1)}$. In other words, as one varies the Kähler form ω in the Kähler class $[\widehat{\omega}]$ (by a change of u), φ is constrained to vary in such a manner which results in the closed equivariant form $\widehat{\omega}+\widehat{\varphi}$ changing by an equivariantly exact form, and hence its integral must be constant. This shows the constancy of the functional from the equivariant coohomology viewpoint. A discussion of Kähler class invariance of related integrals based on just such a variation of a closed equivariant form appears in [7, paragraph after (17)].

If both f and h are the identity functions ($f(x)=x$, $h(z)=z$), then the Euler-Lagrange equation of S is obeyed by metrics g for which φ is a holomorphy potential. By the definition of φ, this holds tautologically for every metric in the variation. Hence S is again a Kähler class invariant. Combining this case with the previous one yields a proof of the invariance of the Futaki invariant, which, for a holomorphic vector field X with a nonempty zero set and holomorphy potential φ, is given by $\mathcal{F}_{[\omega]}(X) = \int_M (s-s_0)\varphi\,\omega^n$, with s_0 denoting the average scalar curvature. Related equivariant expressions for these invariants appear in [9, 7].

2.3.2. The nontrivial case

We continue with the above notations for the case $k=1$. For either $X=0$ or h a nonzero constant function, the Euler-Lagrange requirement is for $f'(s)$ to be a holomorphy potential. This is perhaps the most direct generalization of Calabi's result, which is just the case $f(x)=x^2$.

The main situation studied in this work is the case where neither f nor all the functions h_i are constant, at least one of them is not the identity function, and some $X_i \neq 0$. As mentioned in the introduction, in such a situation it is natural to make a distinction regarding the result of the variation in Proposition 2.2. Namely, we can examine whether or not, for the vector field X_{k+1} having holomorphy potential φ_{k+1} and the fixed vector fields $X_1,\dots X_k$, the set $\{X_1,\dots X_{k+1}\}$ is *linearly independent*.

Consider first for $k=1$, the case where X_2 is a multiple of $X_1 := X$. This translates into the equation $f'(s)h(\varphi)-a\varphi = \text{constant}$ for a constant a, i.e. a functional relation between s and φ, involving an arbitrarily chosen function h. Rather than dealing with this equation directly, we consider Kähler metrics for which $s=H(\varphi)$ for some smooth function $H:\mathbb{C}\to\mathbb{R}$. These can be fitted into the variational framework we have been studying. In fact, one can choose a functional $S_{f,h}$ for which a metric satisfying $s=H(\varphi)$, for a fixed function H as above, is critical. In fact, choose f to be the exponential function (or any function whose derivative is nowhere zero), and then define $h(z) := z/f'(H(z))$. This yields the functional

$$S = \int_M \exp\left[s-H(\varphi)\right]\varphi\,\omega^{\wedge m}$$

whose critical metrics are required to have the integrand as holomorphy potential. If a metric satisfies $s=H(\varphi)$, that integrand equals φ, which is of course a holomorphy potential, so that such a metric is indeed critical, and the resulting "new" vector field X_2 is just X.

Note that in the examples in the next paragraph, the critical metric has φ as a Killing potential for X (and hence chosen to be real valued), and not just a holomorphy potential. Hence the equation $s=H(\varphi)$ need only hold for $H:\mathbb{R}\to\mathbb{R}$. For the purpose of the variation, one can then extend the domain of H to $\mathbb{C}$ in an arbitrary (smooth) fashion.

As mentioned in the introduction, at least for a large class of smooth functions H (aside from the identity, and with φ nonconstant), metrics for which $s=H(\varphi)$, do exist on compact Kähler manifolds. A class of such metrics that will prove quite useful, is the class of SKR metrics, i.e. metrics admitting a special Kähler-Ricci potential, in complex dimension at least three (see [3, Lemma 11.1] and [4]). A precise definition will be given in §3.1. For these metrics, H cannot be completely arbitrary, as it depends on certain boundary conditions (see again §3.1).

Finally we come to the case studied in the next section, where $\{X_1, \ldots X_{k+1}\}$ is a linearly independent set. In §3.1 we will give a construction of such metrics on certain product manifolds, using products of SKR metrics, which works for particular functions f, h_i. In §3.2 we prove, roughly, for $k = 1$ and with a dimensional assumption, that one can produce no such examples from a certain class of SKR metrics.

3. Questions of existence

3.1. Existence

3.1.1. The general construction

Let (M_i, g_i) be a compact Kähler manifold with scalar curvature s_i, $i = 1, \ldots k+1$. Suppose further that each g_i admits a nontrivial Killing vector field X_i with Killing potential φ_i. Assume now that

$$s_i = H_i(\varphi_i) \text{ for some smooth functions } H_i : \mathbb{C} \to \mathbb{R},\ i = 1, \ldots k \text{ and } s_{k+1} = \log \varphi_{k+1}. \tag{3.1}$$

Then the scalar curvature s of the product metric $g = g_1 + \ldots + g_{k+1}$ on the product manifold $M = M_1 \times \ldots \times M_{k+1}$ satisfies $s = s_1 + \ldots + s_k + s_{k+1} = \sum_{i=1}^{k} H_i(\varphi_i) + \log \varphi_{k+1}$, so that

$$\varphi_{k+1} = e^s \prod_{i=1}^{k} e^{-H_i(\varphi_i)}. \tag{3.2}$$

As φ_{k+1}, and in fact each φ_i, may be regarded as a holomorphy potential on M (after pull-back), it follows that

$$g \text{ is critical for } S := \int_M e^{s - \sum_{i=1}^{k} H_i(\varphi_i)}\, \omega^{\wedge m}, \quad m = \dim_{\mathbb{C}} M.$$

In other words, Equation (2.5) is satisfied and defines a holomorphy potential on M for $f(x) = \exp(x)$ and $h_i(z) = \exp(-H_i(z))$. As all vector fields arising from different factors of M and are nontrivial, it is clear that X_{k+1} is not linearly dependent on the X_i, $i = 1, \ldots k$.

The question of existence of such examples is thus reduced to that of constructing the metrics $g_i, i = 1, \ldots k+1$. Constructing g_i, $i = 1, \ldots k$, each satisfying one of the first k conditions in (3.1) can be achieved, for many functions H_i, using SKR metrics on compact manifolds (with the domain of H_i enlarged to $\mathbb{C}$ as in §2.3.2). We complete the demonstration of existence by showing that g_{k+1} can also be constructed using such metrics.

3.1.2. A distinguished SKR metric

An SKR metric is defined as a Kähler metric g which admits a Killing potential τ such that at each τ-noncritical point, all nonzero vectors orthogonal to the complex span of $\nabla\tau$ are eigenvectors of both the Ricci tensor and the Hessian of τ, considered as operators. The term SKR comes from "special Kähler-Ricci", a reference to the fact that for such metrics the Hessian of τ satisfies on an open set an equation involving the Ricci tensor and the Kähler metric. The term "special" refers to the fact that the coefficients in this equation are not arbitrary, as they are functions of τ.

We use a number of results from the classification of SKR metrics [3, 4]. First, the global classification yields two families on compact manifolds. One family, on which we will focus in this subsection, is defined on the projectivization of a line bundle over a Kähler manifold, which is necessarily Einstein unless it is of complex dimension one. The other one is defined on $\mathbb{CP}^m$, and will be considered in §3.2.

To define metrics in the first family, let (N, h) be a Kähler manifold, chosen so that it is also Einstein if $m - 1 = \dim_{\mathbb{C}} N > 1$. Let Q be a smooth function $Q(\tau)$ on a τ-interval $[\tau_{min}, \tau_{max}]$, which is positive on the interior of this interval, and at the endpoints satisfies boundary conditions (see [4, Equation (5.1)])

$$Q(\tau_{min}) = Q(\tau_{max}) = 0, \quad Q'(\tau_{min}) = -Q'(\tau_{max}) \neq 0. \tag{3.3}$$

Choose a complex line bundle $\pi : L \to N$ with a hermitian metric $\langle \cdot, \cdot \rangle$, whose associated metric connection has curvature $\Omega = -2a\omega^{(h)}$, where $\omega^{(h)}$ is the Kähler form of h and a is a constant equal to the value of $Q'/2$ at one of the endpoints. Let $r(\tau) : (\tau_{min}, \tau_{max}) \to (0, \infty)$ be a C^∞ diffeomorphism satisfying $dr/d\tau = ar/Q$, and fix a constant c smaller than τ_{min}. Define a metric on $L \setminus \{$ zero section$\}$ by

$$g = 2(\tau(r) - c)\pi^* h \text{ on } \mathcal{H}, \quad g = Q(\tau(r))/(ar)^2 \, Re\langle \, , \rangle \text{ on } \mathcal{V}. \tag{3.4}$$

Here $\mathcal{V}$ is the vertical distribution in TL while $\mathcal{H}$ the horizontal distribution induced by the connection, and $\mathcal{V}$, $\mathcal{H}$ are declared g-orthogonal. Also, $Re\langle \, , \rangle$ denotes the real part of the hermitian metric while $r : L \to [0, \infty)$ is its norm function, so that $\tau = \tau(r)$, the inverse of $r(\tau)$, becomes a function on L. The metric g and the function τ extend smoothly to the projective compactification of L [4, Section 5].

As mentioned in the previous section, for an SKR metric in complex dimension at least three, the scalar curvature s is a function of the Killing potential τ [3, Lemma 11.1]. We will prove existence of an SKR metric satisfying the condition

$$s = \log(\tau), \tag{3.5}$$

by using it to construct an appropriate function $Q(\tau)$, along with an interval $[\tau_{min}, \tau_{max}]$ on which positivity of Q on its interior and the boundary conditions (3.3) hold. The required metric will be given via Formula (3.4) using this Q and appropriate constants a, c.

We determine Q using various relationships available to us from the theory of SKR metrics. Specifically, given any SKR metric, following [3, Formulas (7.4)], we denote by μ and λ the eigenvalues of the Ricci tensor on $\mathcal{V} = span_{\mathbb{C}}(\nabla\tau)$ and its orthogonal complement $\mathcal{H}$, respectively. We also denote by ϕ the $\mathcal{H}$-eigenvalue of the Hessian of τ. As we take the complex dimension m to be at least three, the functions μ, λ, ϕ are functionally dependent on τ (see again [3, Lemma 11.1]). Finally, the formula $c = \tau - Q/(2\phi)$ defines a constant for any SKR metric with $\phi \neq 0$ (see [3, Lemma 10.1]). The following relations hold among these quantities:

$$\begin{aligned} &(i) && s = 2\mu + 2(m-1)\lambda, \\ &(ii) && Q \, d\lambda/d\tau = 2(\mu - \lambda)\phi \text{ if } m > 2, \\ &(iii) && \mu = -(m+1)\phi' - (\tau - c)\phi'', \\ &(iv) && Q = 2(\tau - c)\phi. \end{aligned} \tag{3.6}$$

All these relations appear in [3]: see Sections 10 and 11 for (i),(ii),(iv) and Section 20 for (iii).

We now proceed as follows: inserting Equation (3.6)(iv) in (3.6)(ii) one gets, after cancellation $(\tau - c)d\lambda/d\tau = \mu - \lambda$. Replacing μ in this equation by $-(m-1)\lambda + (\log\tau)/2$ (an expression obtained from combining (3.5) and (3.6)(i)), we get an ODE for λ, which we solve. Inserting this solution in the equation obtained by equating $-(m-1)\lambda + (\log\tau)/2$ this time with the expression for μ in (3.6)(iii), we obtain an ODE for ϕ. Its solution yields Q using (3.6)(iv). To simplify the expressions involved, we only give Q in the case where $m = 3$ and $c = 0$:

$$Q = -2\tau \left((1/24) \log(\tau) \, \tau - (7/288) \, \tau + A\tau^{-2} + (1/3) \, B\tau^{-3} - C \right)$$

where A, B and C are constants. Next, given interval endpoints τ_{min}, τ_{max}, the three equations in (3.3) give conditions from which A, B and C can be determined. For example, for $\tau_{min} = 1$, $\tau_{max} = 2$ we have

$$A = \frac{20}{33} \log(2) - \frac{49}{132}, \quad B = -\frac{14}{11} \log(2) + \frac{115}{132}, \quad C = \frac{2}{11} \log(2) - \frac{37}{352}.$$

As the function Q, with the above choices of A, B and C, satisfies the boundary conditions, it remains to check whether it is positive in $(\tau_{min}, \tau_{max}) = (1, 2)$. We do so using elementary calculus. First, in this interval $Q > 0$ exactly when $\phi > 0$ (see (3.6)(iv)). Writing $\phi = \alpha\tau + \beta\tau^{-3} + \gamma - \delta\tau\ln(\tau) - \epsilon\tau^{-2}$, where all coefficients are positive, we have

$$\tau^5\phi'' = 12\beta - \delta\tau^4 - 6\epsilon\tau,$$

which is decreasing. One checks that at $\tau = 1$ this expression, i.e. $12\beta - \delta - 6\epsilon$, is negative, to conclude that $\phi'' < 0$, i.e. ϕ' is strictly decreasing on our interval. Checking that $\phi'(1) > 0$ and $\phi'(2) < 0$, while of course $\phi(1) = \phi(2) = 0$ shows that $\phi'(\tau)$ is positive until a unique critical point in $(1, 2)$ on which ϕ is positive. For larger τ the function ϕ decreases, so that for τ between the critical point and the zero at $\tau = 2$, we also have $\phi(\tau) > 0$.

Thus Q is shown to be a proper ingredient for the construction of an SKR metric satisfying (3.5), using Formula (3.4). The other objects needed are the constant c, which is zero in the example above (and so smaller than $\tau_{min} = 1$), and the nonzero constant a chosen to equal the value of $Q'/2$ at one of the endpoints, say at $\tau = 1$. All together this produces via (3.4) a metric on a $\mathbb{CP}^1$-bundle over a complex surface (as $m = 3$ in our example) admitting a Kähler-Einstein metric. This metric can be taken to be g_{k+1} in the construction leading to Equation (3.2), with $\varphi_{k+1} = \tau$.

3.2. Nonexistence

3.2.1. Prologue

Let (M, g) be a compact Kähler manifold with scalar curvature s, and assume there is a Killing potential φ_0 for which $s = H(\varphi_0)$ for a smooth function $H : \mathbb{R} \to \mathbb{R}$ (for instance, g could be an SKR metric). Suppose one wants to examine whether g *also* satisfies the requirement (1.2), namely that $\varphi_{k+1} := p(s)h_1(\varphi_1) \cdot \ldots \cdot h_k(\varphi_k)$ is a holomorphy potential, for holomorphy potentials φ_i and some smooth functions p and h_i, $i = 1 \ldots k$. Now φ_{k+1} can be rewritten in the form

$$\varphi_{k+1} = h_0(\varphi_0)h_1(\varphi_1) \cdot \ldots \cdot h_k(\varphi_k), \tag{3.7}$$

where $h_0 = p \circ H$. Applying the $\bar{\partial}$ operator to both sides of this equation, and then taking metric duals gives, for the corresponding vector fields X_i, $i = 0, \ldots k+1$,

$$X_{k+1} = \sum_{i=0}^{k} a_i X_i, \qquad a_i = h_i'(\varphi_i) \prod_{\substack{\ell=0 \\ \ell \neq i}}^{k} (h_\ell(\varphi_\ell)).$$

Now suppose X_i, $i = 0, \ldots k$ form a basis of the Lie algebra of holomorphic vector fields with a nonempty zero set, and additionally are pointwise linearly independent somewhere (and hence, by analyticity, in an open dense set). Then a necessary condition for X_{k+1} to also be holomorphic is that the a_i, $i = 0, \ldots k$ are all constant. This implication places severe restrictions on the possible functions h_i, $i = 0, \ldots k$ for which (3.7) can be solved.

Theorem 3.1 below describes another situation where Equation (3.7) (or (1.2)) has only a restricted type of solution.

3.2.2. Main theorem

We consider metrics on a hermitian vector space $(V, \langle\, , \rangle)$ (excluding 0) of the form

$$g = S(r)Re\langle\, , \rangle_{\mathcal{V}} + T(r)Re\langle\, , \rangle_{\mathcal{H}}. \tag{3.8}$$

for arbitrary smooth and positive functions S and T of $r = |x|$. Here $\mathcal{V}$ is the distribution in the tangent bundle TV, consisting at $x \neq 0$ of all multiples of x, and $\mathcal{H}$ is its orthogonal distribution. The symbol $Re\langle\,,\rangle_{\mathcal{V}}$ denotes the restriction to $\mathcal{V}$ of the real part of the hermitian metric, and similarly for $Re\langle\,,\rangle_{\mathcal{H}}$.

Among the metrics on $V \setminus \{0\}$ given by (3.8), one can find SKR metrics giving rise to the second of the two families on compact manifolds mentioned in §3.1. For such metrics

$$S(r) = Q(\tau(r))/(ar)^2, \quad T(r) = 2|\tau(r) - c|/(|a|r^2) \tag{3.9}$$

(see [4, Section 6]). Here c and $a \neq 0$ are constants, and the formula involves a smooth positive function $Q(\tau)$ on a given τ-interval (τ_{min}, τ_{max}). It is required that this interval does not contain c. Given Q, the relationship between the potential τ and $r = |x|$ is given again via the equation $dr/d\tau = ar/Q$. As in §3.1, this defines $r(\tau) : (0,\infty) \to (\tau_{min}, \tau_{max})$, with inverse $\tau(r)$.

In Theorem 3.1 below, we write T_r for the derivative of $T(r)$. We also use the notation

$$F = F_A := \langle Ax, x\rangle, \text{ with } A \text{ a } \textit{self-adjoint} \text{ linear operator on } V. \tag{3.10}$$

Theorem 3.1. *Let g be a Riemannian metric on $V \setminus \{0\}$ given by* (3.8)*, having scalar curvature s, and assume A, F are as in* (3.10)*.*

1. *The following are equivalent:*
   (a) *g is Kähler.*
   (b) *g is an SKR metric.*
   (c) *The linear ODE $S = T + (r/2)T_r$ holds.*
   (d) *All linear vector fields $x \to Ax$ (with A self-adjoint) are g-gradient with holomorphy potentials given by*

$$\varphi = (1/2)FT + C, \quad C \text{ constant.} \tag{3.11}$$

2. *If $\dim V \geq 3$ while any of the conditions in Part* (1) *holds, and g satisfies the equation*

$$\varphi_2 = p(s)h(\varphi_1)$$

   for holomorphy potentials φ_1, φ_2 of linearly independent (self-adjoint) linear vector fields and smooth functions p, h, then h is constant.

Remark 3.2 Part (2) of this theorem states that under certain assumptions, no further nontrivial condition of the form given in (1.2) for $k = 1$ holds, with φ_2 and φ_1 holomorphy potentials for linearly independent vector fields. □

Remark 3.3 Under certain conditions some of the SKR metrics in Theorem 3.1 extend to $\mathbb{CP}^m$, with V viewed as embedded in it in the usual way (see [4, Section 6]). Namely, aside from positivity on a given τ-interval (τ_{min}, τ_{max}), the function Q is required to satisfy the same boundary conditions as in (3.3):

$$Q(\tau_{min}) = Q(\tau_{max}) = 0, \quad Q'(\tau_{min}) = -Q'(\tau_{max}) \neq 0. \tag{3.12}$$

Furthermore, a must equal the value of $Q'/2$ at one of the endpoints, while the constant c must equal one of the endpoint values τ_{min} or τ_{max}.

Note that although the underlying space for Theorem 3.1 is a (noncompact) vector space of dimension m, for those SKR metrics that extend to $\mathbb{CP}^m$, Part (2) holds on the (compact) complex projective space. This follows since the above linear vector fields also extend to give all gradient holomorphic vector fields on $\mathbb{CP}^m$. (Note that as $\mathbb{CP}^m$ is is simply connected, composition with the standard almost complex structure provides an isomorphism between gradient holomorphic vector fields and Killing fields.) □

The proof of this theorem will be established in a series of propositions.

Proposition 3.4. *(a) and (c) in Part (1) of Theorem 3.1 are equivalent.*

Proof. Suppose g is a metric as in (3.8), with Kähler form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$, where J is the standard complex structure on V. Consider

$$3d\omega(v_0, v_1, v_2) = d_{v_0}(\omega(v_1, v_2)) - d_{v_1}(\omega(v_0, v_2)) + d_{v_2}(\omega(v_0, v_1)) \\ - \omega([v_0, v_1], v_2) + \omega([v_0, v_2], v_1) - \omega([v_1, v_2], v_0).$$

Using the antisymmetry of $d\omega$ one sees that it is enough to check closedness for ω at a point using triples chosen from a basis of tangent vectors, such that the triple contains no repetition, and only one choice of order of the vectors needs to be examined. We use a basis induced (generically) by the mutually orthogonal vector fields $v = rd/dr$, $u = Jv$ and horizontal vector fields (sections of $\mathcal{H}$) w, w', w'' which also commute with v and u (in other words, they are projectable as well as horizontal). The symmetry considerations just mentioned mean that one need only check closedness on the triples $\{u, v, w\}$, $\{w, w', w''\}$, $\{u, w, w'\}$ and $\{v, w, w'\}$. We now carry out these calculations.

For the first triple, orthogonality as well as the commutation relations imply that $d\omega(v, u, w) = d_w(\omega(v, u)) = d_w(g(u, u)) = d_w(S(r)Re\langle u, u\rangle) = 0$ as $d_w r = 0$.

For the next triple, we use the standard submersion $\pi : V\setminus\{0\} \to \mathbb{P}(V) = \mathbb{CP}^{m-1}$. This map, with V equipped with g and $\mathbb{P}(V)$ with the Fubini-Study metric h, is a horizontally homothetic submersion, that is, $\pi^*h = g/f$ for a positive valued function f on $\pi : V \setminus \{0\}$ whose gradient is vertical, i.e. is in $\mathcal{V}$ (in fact, $f(x) = |x|^2$). To compute, we denote the Kähler form of h by ω^h. Also, note that the horizontal lift operation takes the bracket of two vector fields w, w' on $\mathbb{P}(V)$ to $[w, w']^{\mathcal{H}}$ on $V \setminus \{0\}$ (where w, w' each denote both a vector field on $\mathbb{P}(V)$ and its horizontal lift with respect to π (see [3, Remark 14.1]). Therefore, a directional derivative term in the expression for $d\omega(w, w', w'')$ equals, for example, $d_w(\omega(w', w'')) = d_w(f\omega^h(w', w'')) = f(d_w(\omega^h(w', w'')))$, due to the verticality of ∇f. On the other hand, a term involving the Lie brackets equals, for example, $\omega([w, w'], w'') = \omega([w, w']^{\mathcal{H}}, w'') = f\omega^h([w, w'], w'')$. Thus altogether $d\omega(w, w', w'')$ equals f times $d\omega^h(w, w', w'')$, which vanishes as ω^h is closed.

Next, by orthogonality and commutation relations, $3d\omega(u, w, w') = d_u(\omega(w, w') - \omega([w, w'], u)$. Regard now V as the total space of a line bundle over $\mathbb{P}(V)$. The hermitian metric $\langle\cdot,\cdot\rangle$ induces a hermitian fiber metric on this line bundle, whose curvature two form is a multiple of ω^h. One deduces from this the relation

$$[w, w']^{\mathcal{V}} = -2\omega^h(w, w')u. \tag{3.13}$$

It follows that $\omega([w, w'], u) = g(J[w, w']^{\mathcal{V}}, u) = -2g(-\omega^h(w, w')u, -v) = -2\omega^h(w, w')g(u, v) = 0$ as u, v are g-orthogonal. On the other hand $d_u(\omega(w, w')) = d_u(g(Jw, w')) = d_u(TRe\langle Jw, w'\rangle) = 0$ as u is Killing and $d_u r = 0$.

Finally, as in the previous case, commutation relations and orthogonality give $3d\omega(v, w, w') = d_v(\omega(w, w')) - \omega([w, w'], v)$. The first term on the right hand side is

$$\begin{aligned} d_v(g(Jw, w')) &= r\frac{d}{dr}(TRe\langle iw, w'\rangle_{\mathcal{H}}) \\ &= rT_r Re\langle iw, w'\rangle + T(Re\langle i\nabla_v w, w'\rangle + Re\langle iw, \nabla_v w'\rangle) \\ &= rT_r Re\langle iw, w'\rangle + T(Re\langle \nabla_{iw} v, w'\rangle + Re\langle iw, \nabla_{w'} v\rangle) \\ &= (rT_r + 2T)Re\langle iw, w'\rangle \end{aligned}$$

where we have used in the last line the fact that ∇v, treated as a bilinear form, is the Hessian of $r^2/2$, which is of course $Re\langle\cdot,\cdot\rangle$.

As for the second term in right hand side, we again use relation (3.13) to give

$$\begin{aligned}\omega([w,w'],v) &= g(J[w,w']^{\mathcal{V}},v) = -2g(-\omega^h(w,w')u,u) = 2\omega^h(w,w')g(u,u)\\ &= 2\omega^h(w,w')SRe\langle u,u\rangle_{\mathcal{V}} = 2[Re\langle Jw,w'\rangle/Re\langle v,v\rangle]SRe\langle u,u\rangle = 2SRe\langle iw,w'\rangle,\end{aligned}$$

where in the second to last equality we have used the definition of the Fubini-Study Kähler form, and in the last line we used the equality $Re\langle v,v\rangle = Re\langle u,u\rangle$.

Thus $d\omega(v,w,w') = 0$ if and only if $rT_r + 2T = 2S$, i.e. the equation in Part (1)(c) of Theorem 3.1 holds. This completes the proof of equivalence. □

Next, we have

Proposition 3.5. *Conditions (b) and (c) of Part (1) of Theorem 3.1 are equivalent.*

Proof. An SKR metric is in particular Kähler, so satisfies (c) by Proposition 3.4. For the converse, suppose (c) holds for some smooth and positive functions $S(r)$, $T(r)$. Fix constants $a>0$, c and define $\tau(r) := ar^2T/2 + c$ and then $Q := ar\tau'(r)$. Note that $\tau > c$ for $r > 0$. We compute $T + (r/2)T_r = 2(\tau - c)/(ar^2) + (r/2)[-4(\tau - c)r^{-3}/a + 2r^{-2}\tau'/a] = \tau'/(ar) = Q/(ar)^2$, so that $S = Q/(ar)^2$, hence S and T are of the form given in Equation (3.9) for the coefficients of an SKR metric. Also, $\tau' = arT + ar^2T_r/2 = ar(T + (r/2)T_r) = arS > 0$, so that $Q > 0$ (if $r > 0$) as required for an SKR metric. Finally, by Proposition 3.4, the metric g given by (3.8) (with (3.9)) is Kähler. Hence g is an SKR metric. □

In the course of proving that (c) and (d) in Part (1) of Theorem 3.1 are equivalent, we consider a more general family of metrics. Here T_r will denote a partial derivative of a function $T(r,F)$.

Proposition 3.6. *Let S:=S(r,F), T:=T(r,F) be positive C^1 functions on a domain in $\mathbb{R}^2$. Consider a metric defined on $V \setminus \{0\}$ for a vector space V, of the form*

$$g = S(r,F)Re\langle\,,\rangle_{\mathcal{V}} + T(r,F)Re\langle\,,\rangle_{\mathcal{H}},$$

with $r = |x|$ and F (and A) as in (3.10) but with A not a multiple of the identity. The linear PDE

$$2[(S-T)F]_F = r\,T_r \tag{3.14}$$

holds if and only if the linear vector field $x \to Ax$ is gradient with respect to g and has holomorphy potential

$$\varphi = (1/2)\int T dF + C(r), \quad \textit{C(r) an arbitrary } C^1 \textit{ function.} \tag{3.15}$$

Note that if the metric dependence on F is nontrivial, the above vector field is the only gradient linear one.

Proof. Consider the orthogonal splitting $Ax = (F/r^2)x + y$, with $y \in \mathcal{H}$. The one-form P dual to $x \to Ax$ is given by

$$\begin{aligned}P = g(Ax,\cdot) &= (Fr^{-2})S\cdot Re\langle x,\cdot\rangle + T\cdot Re\langle Ax - (Fr^{-2})x,\cdot\rangle\\ &= Fr^{-2}(S-T)Re\langle x,\cdot\rangle + T\cdot Re\langle Ax,\cdot\rangle,\end{aligned}$$

so that

$$P = r^{-1}F(S-T)\,dr + (T/2)\,dF, \tag{3.16}$$

where we have used the easily verifiable relations

$$Re\langle x,\cdot\rangle = r\,dr,\quad Re\langle Ax,\cdot\rangle = (1/2)dF,$$

with the latter relation holding precisely since A is self-adjoint. We compute

$$dP = r^{-1}[F(S-T)]_F\,dF\wedge dr + (1/2)\,T_r\,dr\wedge dF.$$

As A is not a multiple of the identity, this expression vanishes exactly when (3.14) holds. Writing $P = d\varphi = \varphi_r\,dr + \varphi_F\,dF$ together with (3.16) gives (3.15) after integrating φ_F with respect to F. Note that one can also arrive at the formula for the holomorphy potential from (3.14). Namely, the coefficient of dr in (3.16) is, up to the factor $2/r$ just the F-integral of the left hand side of (3.14), so that one can equate φ_r with $2/r$ times the F integral of the right hand side of (3.14) and then integrate with respect to r while switching the order of integration (with respect to F and r). □

The proof of the equivalence of (c) and (d) in Part (1) of Theorem 3.1 now follows at once, for linear vector fields for which A is not a multiple of the identity. In fact, the linear ODE in (c) follows immediately from (3.14) and the independence from F of S and T. As F does not appear in this ODE, the conclusion that $x\to Ax$ is gradient holds for *any* linear vector field with A as in (3.10) (but not a multiple of the identity). The expression (3.11) for the potential is a consequence of (3.15) and has $C(r) = C$ constant, as one can see by using the ODE in (c) in the equation relating φ_r with the coefficient of dr in (3.16), and then integrating with respect to r.

To complete the proof that (c) and (d) of Part (1) of the theorem are equivalent, it is enough to show that (c) implies formula (3.11) for the holomorphy potential, in the case where A *is* a multiple of the identity. In fact, a computation as the one leading to (3.16) and the material past it shows in this case that $d\varphi = P = \varphi_r\,dr$, so that φ is a function of r. Now the special Kähler-Ricci potential $\tau = \tau(r)$ is also a holomorphy potential which is a function of r. But the holomorhpic vector fields associated with these two potentials must be linearly dependent. This can be seen from equating the term $\bar\partial r$ in the relation $\imath_X\omega = \bar\partial\varphi = \bar\partial(\varphi(r)) = \varphi'(r)\bar\partial r$ with the same term in the analogous relation for $\tau(r)$ (with ω denoting the Kähler form of g). This gives, after taking the dual, that the corresponding $(1,0)$-vector fields are pointwise proportional, so their ratio is real-valued, yet must be a holomorphic function, hence is constant. Thus $v = \nabla\tau$ is a linear vector field for which the corresponding matrix A is a multiple of the identity. But for τ we have, as (b) implies (c) in Part (1), that (with I denoting the identity operator) $\tau = ar^2T/2 + c = \langle aIx, x\rangle T/2 + c = (1/2)FT/2 + c$ for constants a and c (see the proof of Proposition 3.5). In other words, formula (3.11) holds for τ and hence also for any other holomorphy potential for a linear vector field for which A is a multiple of the identity.

To continue with Part (2) of Theorem 3.1, we need the following lemma, in which I again stands for the identity operator on a vector space V.

Lemma 3.7. *Let A and B be two self-adjoint operators on a vector space V with $\dim(V)\geq 3$. If y, Ay and By are linearly dependent for each $y\in V$, then A, B, and I are linearly dependent in $Hom(V,V)$.*

Proof. Clearly if A or B are a multiple of I, the conclusion holds. Hence we assume A is not a multiple of the identity. Choose eigenvectors u, v of A with distinct eigenvalues λ, μ, respectively. For any scalar $t\neq 0$, the vectors $u+tv$, $A(u+tv) = \lambda u + t\mu v$, $B(u+tv)$ are linearly dependent by assumption, while the first two of these vectors are linearly independent (their coefficient determinant $t(\mu-\lambda)$ is nonzero). Hence, by the assumption of the lemma, for some scalars $a(t)$, $b(t)$ we have $Bu + tBv = a(t)(u+tv) + b(t)(\lambda u + t\mu v) = (a(t)+b(t)\lambda)u + t(a(t)+b(t)\mu)v$. Differentiating with respect to t gives

$$Bv = \alpha(t)u + \beta(t)v \tag{3.17}$$

for appropriate coefficients $\alpha(t)$, $\beta(t)$.

We now make the following observation: *An A-eigenvector v which is not a B-eigenvector belongs to an A-eigenspace of codimension at most one.* In fact, otherwise let u, w be two linearly independent A-eigenvectors not in the A-eigenspace of v. The two pairs $\{v, u\}$ and $\{v, w\}$ both satisfy an equation of the form (3.17). If v is not a B-eigenvector then the coefficient $\alpha(t)$ and the corresponding one for w are both nonzero. Equating the right hand sides of the two equations together implies that $\{u, v, w\}$ are linearly dependent. This is a contradiction to the fact that these vectors do not all belong to the same A-eigenspace, and the two of them that may be in the same A-eigenspace are linearly independent (note that the coefficient of v in the linear dependence equation cannot be zero or else w and u will be proportional).

From the observation above it follows that *if* $\dim(V) \geq 3$*, every A-eigenvector is also a B eigenvector*. For suppose not. Then, by the previous paragraph, there is a codimension one A-eigenspace which is not a B-eigenspace (recall that A is not a multiple of the identity, so the codimension is in fact equal to one). Consider the one dimensional A-eigenspace orthogonal to this codimension one eigenspace (they are orthogonal as A is self-adjoint). It cannot be also a B-eigenspace, for then by orthogonality the codimension one A-eigenspace will also be a B-eigenspace, contradicting our assumption. Thus, according to the previous paragraph, this one dimensional A-eigenspace must be of codimension one. This would make $\dim(V) \leq 2$, a contradiction.

Hence, let $\{v_i, i = 1, \ldots \dim(V)\}$ be a basis of eigenvectors for both A and B, with corresponding eigenvalues $\lambda_i^A, \lambda_i^B, i = 1, \ldots \dim(V)$, given in their multiplicity. Consider $v = \sum_{i=1}^{\dim(V)} v_i$. Assume $(aA + bB + cI)v = 0$ for some fixed scalars a, b, c not all zero. This is equivalent to $\sum_{i=1}^{\dim(V)} (a\lambda_i^A + b\lambda_i^B + c)v_i = 0$, or, by the linear independence of the v_i, to a system of linear equations of the form

$$a\lambda_i^A + b\lambda_i^B + c = 0, \quad i = 1 \ldots \dim(V),$$

which is simply equivalent to the diagonal matrix equation $a\langle \lambda_1^A, \ldots \lambda_{\dim(V)}^A \rangle + b\langle \lambda_1^B, \ldots \lambda_{\dim(V)}^B \rangle + cI = 0$, so $aA + bB + cI = 0$.

□

We now consider Part (2) of Theorem 3.1. We need to determine whether, for a metric g given by (3.8) and satisfying $S = T + (r/2)T_r$, two (not affinely related) holomorphy potentials φ_1, φ_2 (of linear vector fields) may be related by

$$\varphi_2 = p(s)h(\varphi_1),$$

where s is the scalar curvature of g and p, h are arbitrary smooth functions. Since metrics as in (3.8) are of cohomogeneity one, it is known that the scalar curvature is a function of the norm r (this can also be seen as g is an SKR metric, hence s is a function of τ, which is a function of r). It is thus sufficient to replace p(s) by a function $u(r)$, giving

$$\varphi_2 = u(r)h(\varphi_1). \tag{3.18}$$

Substituting(3.11), Equation (3.18) takes the form

$$T(r)\langle \overline{A}_2 x, x\rangle + C = u(r)h\left(T(r)\langle \overline{A}_1 x, x\rangle + C_1\right),$$

where the indices correspond to those of the potentials, $\overline{A}_i = A_i/2$, $i = 1, 2$ and C_1, C are constants.

In order to understand the possible solutions to this equation, we express it a bit differently. Making the change of variable $y = (T(r))^{1/2}x$, we have $|y|^2 = r^2T(r)$ and so

$$\langle \overline{A}_2 y, y\rangle + C = q(|y|^2)\widehat{h}(\langle \overline{A}_1 y, y\rangle), \tag{3.19}$$

with $q(|y|^2) = u(r)$ and $\widehat{h}(w) = h(w + C_1)$.

We now have

Proposition 3.8. *If* $\dim(V) \geq 3$ *then any self adjoint solutions* $\overline{A}_1$*,* $\overline{A}_2$ *of Equation* (3.19)*, together with the identity element* I*, are linearly dependent.*

Proof. Differentiating (3.19) shows that the gradients of the three quadratic forms appearing in (3.19) are pointwise linearly dependent, i.e. for all $y \in V$

$$\overline{A}_2 y = aIy + b\overline{A}_1 y$$

for functions $a = q'(|y|^2)\widehat{h}(\overline{F}_1)$, $b = \widehat{h}'(\overline{F}_1)q(|y|^2)$, with $\overline{F}_1 = \langle \overline{A}_1 y, y\rangle$. We now use Lemma 3.7. □

Suppose $a_0 I + a_1\overline{A}_1 + a_2\overline{A}_2 = 0$ for constants a_i, $i = 0, 1, 2$ not all zero. If $a_2 = 0$ then A_1 is a multiple of the identity, so that Equation (3.19) has the form $\langle \overline{A}_2 y, y\rangle + C = q(|y|^2)\widehat{h}(m|y|^2)$ for some constant m. Setting $t = |y|^2$ and $z = \langle \overline{A}_2 y, y\rangle$ we have $z + C = q(t)\widehat{h}(t)$, which implies, via separation of variables, that z is constant. But then $\overline{A}_2 = 0$ so φ_2 is a holomorphy potential associated with the zero vector field, which contradicts linear independence of the vector fields. Hence $a_2 \neq 0$, and we write $\overline{A}_2 = kI + l\overline{A}_1$ for constants k, l. We now have from (3.19)

$$k|y|^2 + l\langle \overline{A}_1 y, y\rangle + C = q(|y|^2)\widehat{h}(\langle \overline{A}_1 y, y\rangle).$$

Setting $t = \langle y, y\rangle$, $z = \overline{F}_1 = \langle \overline{A}_1 y, y\rangle$ we have $kt + lz = q(t)\widehat{h}(z)$. taking the partial derivatives with respect to t and z we get

$$k = q'(t)\widehat{h}(z), \quad l = q(t)\widehat{h}'(z).$$

Now $k \neq 0$ as $\overline{A}_2$ and $\overline{A}_1$ represent linearly independent vector fields. If $l \neq 0$, separation of variables now gives that both q and $\widehat{h}$ must be constant, but also that q' and $\widehat{h}'$ are constant. Thus the latter two functions must be identically zero, which contradicts k and l being nonzero. Hence we must have ($k \neq 0$ and) $l = 0$. In this case, as q cannot be zero (since it forces $q' = 0$ and hence $k = 0$), we must have $\widehat{h}' = 0$, i.e. $\widehat{h}$ is constant, and so is h, as required. (Note that in this case q' is a nonzero constant, so q is affine in t. Finally note that in this case $\overline{A}_2$ is special, being a multiple of the identity, which means that the corresponding vector field is a multiple of $\nabla\tau$, with τ the special Kähler-Ricci potential. The resulting equation φ_2 =constant$\cdot p(s)$ corresponds to the characterization of the scalar curvature of an SKR metric as a function of τ.) This completes the proof of Theorem 3.1.

Acknowledgements

The author thanks Andrzej Derdzinski for many discussions that significantly improved both the style and the mathematical content. In particular, the idea behind §3.1.1 and Lemma 3.7 with its application are due to him.

References

[1] Apostolov, V., Calderbank, D.M.J., Gauduchon P.: Hamiltonian 2-forms in Kähler geometry, I. General theory. J. Differential Geom. 73 (2006), 359–412

[2] Calabi, E.: Extremal Kähler metrics. Seminar on Differential Geometry, pp. 259–290, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J., 1982

[3] Derdzinski, A., Maschler, G.: Local classification of conformally-Einstein Kähler metrics in higher dimensions. Proc. London Math. Soc. 87 (2003), 779–819

[4] Derdzinski, A., Maschler, G.: Special Kähler-Ricci potentials on compact Kähler manifolds. J. reine angew. Math. 593 (2006), 73– 116

[5] Kobayashi, S.: Transformation groups in differential geometry. Reprint of the 1972 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995

[6] Liu, K.: Holomorphic equivariant cohomology. Math. Ann. 303 (1995), 125–148

[7] Maschler, G.: Central Kähler metrics. Trans. Amer. Math. Soc. 355 (2003), 2161–2182

[8] Page, D.: A compact rotating gravitational instanton. Phys. Lett. B 79 (1978), 235–238

[9] Tian, G.: Kähler-Einstein metrics on algebraic manifolds. Transcendental methods in algebraic geometry (Cetraro, 1994), 143–185, Lecture Notes in Math., 1646, Springer, Berlin, 1996