METRIC PAIRS AND THE FUTAKI CHARACTER

GIDEON MASCHLER

Department of Mathematics
University of Toronto
Toronto M5S 3G3
Canada
E-mail: maschler@math.toronto.edu

1991 Mathematics Subject Classification: Primary 53C55,
Secondary 53C25, 32L07, 32C17

The non-vanishing of the Futaki character gives an obstruction to the existence of Kähler metrics of constant scalar curvature, having a Kähler form belonging to a fixed Kähler class [6, 4]. It is shown that, in combination with the resolution of the Calabi conjecture [18], one has an analogous obstruction on *pairs* of metrics having Kähler forms belonging to a *fixed pair* of Kähler classes. If the *difference* of the Futaki characters on two classes of fixed total volume does not vanish identically, there cannot exist a pair of metrics, with Kähler forms in these classes, having the same Ricci form *and* the same harmonic Ricci form. When the obstruction vanishes, results in [8] are used to construct non-trivial examples of such pairs, which are also extremal in the sense of Calabi [3].

Partially supported by the Edmund Landau Center for research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany), and by the Max Planck Institute in Bonn (Germany).

1. Introduction

In [5, 6, 4], Futaki defined a character on the Lie algebra of holomorphic vector fields on a compact Kähler manifold. Such a manifold in fact admits a family of characters, one attached to each Kähler class in the second cohomology group. The non-vanishing of one of these characters provides an obstruction to the existence of Kähler metrics of constant scalar curvature (with Kähler form) in the corresponding class.

The purpose of this note is to show that Futaki's characters, interpreted via Yau's solution to the Calabi conjecture, also give information about *pairs* of metrics in distinct Kähler classes. We call a pair of Kähler metrics a *harmonic (Calabi-Yau) pair*, if the two metrics share the same Ricci form *and* the same harmonic Ricci form. Such a condition is, of course, overdetermined. It is of interest, however, that there exists a topological obstruction to its fulfillment. Our results slightly enhance the following.

Main Theorem. *Let M be a compact Kähler manifold, and $\Omega, \tilde{\Omega}$ a pair of Kähler classes having equal total volume. If the difference of the Futaki characters of these two classes does not vanish identically, then there does not exist a harmonic pair of Kähler metrics with Kähler forms $(\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}$. When this obstruction vanishes, there exist examples of harmonic pairs which are also extremal.*

Here an extremal Kähler metric is one which minimizes the L^2-norm of the scalar curvature in its Kähler class (see Calabi [3]). Note that one can easily find examples of harmonic pairs, each of which is a product of metrics of constant scalar curvature, or indeed, a product of extremal metrics. One merit of the examples we give, is that they do not consist of product metrics. To construct them we rely heavily on results of Hwang [8].

After gathering the necessary preliminaries in Section 2, we demonstrate the existence of the obstruction in Section 3, and relate it both to Mabuchi's K-energy map [13] and to extremal metrics. Section 4 is devoted to giving extremal harmonic examples.

Acknowledgements

I wish to thank Claude LeBrun for his guidance in the preparation of my thesis [14]. A chapter of that work contains a sizable portion of this note. I acknowledge the hospitality of the Institute of Mathematics at the Hebrew University of Jerusalem, where the majority of the preparatory work was carried out.

2. Preliminaries

2.1. **The Futaki Invariant.** Let $M := M_n$ be a compact Kähler manifold of complex dimension n. Given a Kähler metric g on M, with Kähler form ω and Ricci form ρ, denote by ρ_H the harmonic part of ρ. Since ρ and ρ_H belong to the same cohomology class, there exists (cf. [7, Chapter 1, Section 2]) a smooth real valued function F, called the **Ricci potential**, such that

$$\rho - \rho_H = i\partial\bar{\partial}F. \tag{1}$$

Unless otherwise stated, we normalize F to be L^2-perpendicular to the constants.

To recall the definition of the Futaki invariant, we denote by $h(M)$ the Lie algebra of holomorphic vector fields on M.

Definition. Let (M_n, ω) be a compact Kähler manifold with Ricci potential F. The **Futaki character** is the map $\mathcal{F}_{[\omega]} : h(M) \to \mathbb{C}$ given by

$$\mathcal{F}_{[\omega]}(\Xi) = \int_M \Xi F \frac{\omega^{\wedge n}}{n!}. \tag{2}$$

Remark. The values of this character do not depend on the choice of metric in the Kähler class $[\omega]$ (see [1, 4, 6]), i.e., it is a Kähler class invariant. This invariance, in fact, implies that $\mathcal{F}_{[\omega]}$ is a Lie algebra character (cf. [4]).

2.2. **Scalar Curvature, Extremal Kähler metrics.** The relation between the scalar curvature s and the Ricci potential F, is arrived at by taking traces with respect to ω in equation (1), and then applying the Hard Lefschetz Theorem. One has[a]:

$$\Delta F = s - s_0, \qquad s_0 := \left(\int_M s\frac{\omega^{\wedge n}}{n!}\right) \Big/ \left(\int_M \frac{\omega^{\wedge n}}{n!}\right). \tag{3}$$

Note that s_0, the average value of the scalar curvature over M, depends only on the Kähler class. It follows from (1) and (3) that a Kähler metric has constant scalar curvature if and only if its Ricci form is harmonic.

The scalar curvature figures in Calabi's notion of an extremal Kähler metric through its L^2 norm [3].

Definition. A Kähler metric g with Kähler form $\omega := \omega_g$ on a complex manifold M_n will be called an **extremal** Kähler metric if it is a critical point of the functional $g \longrightarrow \int_M s_g^2 \frac{\omega_g^{\wedge n}}{n!}$, considered over Kähler metrics (with Kähler forms) in the class $[\omega]$. Here s_g denotes the scalar curvature of g.

At times one has an alternative characterization in terms of vector fields. A smooth complex valued function f will be called a **holomorphy potential** if the $(0,1)$-form $\bar{\partial} f$ is metrically dual to a *holomorphic* vector field $\Xi := \Xi_f$. Such a vector field is called **gradient**. Then for M compact, a Kähler metric is extremal if and only if its scalar curvature is a holomorphy potential, i.e., in local complex coordinates, $s_{,\bar{\beta}}^{,\alpha} = 0$ (Calabi [3]). In particular, if g is a Kähler-Einstein metric, or, more generally, any Kähler metric of constant scalar curvature, it is extremal.

3. The Reflection Character

3.1. **Definition, Invariance.** Recall that on a compact Kähler manifold M, the Ricci form of any Kähler metric is closed, and its corresponding second cohomology class in $H^2(M, \mathbb{C})$ is the first Chern class $c_1(M)$. Also, the Ricci form is completely determined by the complex structure and the volume form, via the formula $2\pi\rho = -i\partial\bar{\partial} \log \det g$. Hence one has the following two equivalent formulations of the Calabi Conjecture, which was resolved by Yau [18].

Theorem 3.1 (Calabi-Yau [18])**.** *Let M_n be a compact Kähler manifold. If $\bar{\rho}$ is a real closed $(1,1)$-form representing $c_1(M)$ (if Ψ is a real non-degenerate (n,n)-form), then in every Kähler class there exists a unique Kähler form ω, whose Ricci form equals $\bar{\rho}$ (whose volume form equals a positive multiple of Ψ).*

[a]For convenience, we have absorbed into s a factor of $-1/2$, as compared with the standard Riemannian definition.

We now define a new character attached to pairs of Kähler classes, and proceed to relate it to the Futaki character.

Definition. Let M_n be a Kähler manifold, Ω and $\tilde{\Omega}$ two Kähler classes. Let ω be a Kähler form in Ω with **Calabi-Yau representative** $\tilde{\omega}$ in $\tilde{\Omega}$, i.e. the unique Kähler form with Ricci form equal to that of ω. Define the **reflection potential** $\Phi := \Phi_{\omega,\tilde{\Omega}}$ of the pair $(\omega, \tilde{\Omega})$ to be the smooth real valued function given up to an additive constant by

$$\rho_H - \tilde{\rho}_H = i\partial\bar{\partial}\Phi,$$

where ρ_H, $\tilde{\rho}_H$ are the ω-harmonic and $\tilde{\omega}$-harmonic representatives, respectively, in the class c_1. The **reflection character** is defined to be the Lie algebra character $\mathcal{R}_{\Omega}^{\tilde{\Omega}} : h(M) \to \mathbb{C}$, given by

$$\mathcal{R}_{\Omega}^{\tilde{\Omega}}(\Xi) = \int_M \Xi(\Phi)\frac{\omega^{\wedge n}}{n!}. \tag{4}$$

We call $(\omega, \tilde{\omega})$ a **Calabi-Yau (metric) pair**, and say they form a **harmonic Calabi-Yau pair**, or simply a harmonic pair, if $\rho_H = \tilde{\rho}_H$.

We proceed at once to show that this invariant is well defined, and indeed, is a character.

Proposition 3.2. *Keeping notations as in the definition, $\mathcal{R}_{\Omega}^{\tilde{\Omega}}$ does not depend on the choice of ω in Ω. Furthermore, we have $\mathcal{R}_{\Omega}^{\tilde{\Omega}} \equiv -A\mathcal{R}_{\tilde{\Omega}}^{\Omega}$, where $A := A_{\tilde{\Omega}}^{\Omega} = \frac{\Omega^{\wedge n}}{\tilde{\Omega}^{\wedge n}}$ (the volume ratio of the classes).*

Proof. Since ω, $\tilde{\omega}$ are Calabi-Yau related, one has, by Theorem 3.1, $A\tilde{\omega}^{\wedge n} = \omega^{\wedge n}$ $(A > 0)$. Also, if F and $\tilde{F}$ are the Ricci potentials of ω and $\tilde{\omega}$, respectively, we have, $i\partial\bar{\partial}\Phi = \rho_H - \tilde{\rho}_H = \rho_H - \rho + \rho - \tilde{\rho}_H = \rho_H - \rho + \tilde{\rho} - \tilde{\rho}_H = -i\partial\bar{\partial}F + i\partial\bar{\partial}\tilde{F}$. Choosing $\Phi = -F + \tilde{F}$, and using the volume form proportionality of the pair, we get

$$\mathcal{R}_{\Omega}^{\tilde{\Omega}}(\Xi) = \textstyle\int_M \Xi(\Phi)\frac{\omega^{\wedge n}}{n!} = -\int_M \Xi(F)\frac{\omega^{\wedge n}}{n!} + \int_M \Xi(\tilde{F})\frac{\omega^{\wedge n}}{n!} =$$

$$-\textstyle\int_M \Xi(F)\frac{\omega^{\wedge n}}{n!} + A\int_M \Xi(\tilde{F})\frac{\tilde{\omega}^{\wedge n}}{n!} = -\mathcal{F}_{\Omega}(\Xi) + A\mathcal{F}_{\tilde{\Omega}}(\Xi),$$

which is an expression depending, for any fixed Ξ, only on the Kähler classes. The last statement follows again from the relation between the volume forms, together with the relation $\Phi_{\omega,\tilde{\Omega}} = -\Phi_{\tilde{\omega},\Omega}$. □

Note that on Kähler classes of metrics of fixed total volume, the reflection character satisfies a cocycle condition:

$$\mathcal{R}_{\Omega}^{\tilde{\Omega}} \equiv -\mathcal{R}_{\tilde{\Omega}}^{\Omega}, \qquad \mathcal{R}_{\Omega}^{\tilde{\Omega}} + \mathcal{R}_{\tilde{\Omega}}^{\hat{\Omega}} \equiv \mathcal{R}_{\Omega}^{\hat{\Omega}}.$$

3.2. **Relation To The K-Energy Map.** Given a Kähler class Ω, fix a Kähler form $\omega_0 \in \Omega$. Let ϕ_t be a one-parameter family of smooth real valued functions such that $\phi_0 = 0$, and $\omega_t := \omega_0 + i\partial\bar{\partial}\phi_t$ is a Kähler form. Denoting $\omega := \omega_1$, Mabuchi's K-energy map [13] is the functional on Kähler forms in Ω, given by

$$\mu(\omega) := \mu_{\Omega,\omega_0}(\omega) := M(\omega_0, \omega) := -\int_0^1 (\int_M \dot{\phi}_t(s_{\phi_t} - s_0)\frac{\omega_t^{\wedge n}}{n!})\mathrm{dt}.$$

Here the upper dot in $\dot{\phi}_t$ denotes differentiation with respect to t, and s_{ϕ_t} is the scalar curvature of ω_t. The functional μ is independent of the path ϕ_t between ω_0 and ω_1, and changes by a constant upon changing the basepoint ω_0, while $M(\_,\ \_)$ satisfies a cocycle condition. The K-energy map has Kähler forms of metrics of constant scalar curvature as critical points.

Working for simplicity with Kähler classes of metrics of fixed total volume, the reflection character can be related to the K-energy map. Fixing $\omega_0 \in \Omega$, if $\tilde{\omega}_0 \in \tilde{\Omega}$ is the Calabi-Yau representative of ω_0 in $\tilde{\Omega}$, define

$$\nu^{\tilde{\Omega}}_{\Omega,\omega_0}(\omega) = \mu_{\Omega,\omega_0}(\omega) - \mu_{\tilde{\Omega},\tilde{\omega}_0}(\tilde{\omega}).$$

Since the derivative of $\mu(\omega)$ along the one-parameter group of diffeomorphisms obtained by exponentiating the real part of a given holomorphic vector field is exactly the real part of the Futaki invariant, evaluated on this vector field, the functional ν has its derivative in the same direction equal to the real part of the reflection character of the two classes, evaluated on the vector field.

3.3. **Relation To Extremal Metrics.**

Proposition 3.3. *Let M be a compact Kähler manifold and Ω, $\tilde{\Omega}$ two Kähler classes (having equal total volume). Suppose $\omega \in \Omega$ is the Kähler form of a Kähler metric g, $\tilde{\omega} \in \tilde{\Omega}$ its Calabi-Yau representative with corresponding metric $\tilde{g}$, $\rho = \tilde{\rho}$ their shared Ricci form, F, $\tilde{F}$ the corresponding Ricci potentials and Φ the reflection potential of $(\omega, \tilde{\Omega})$. We will assume as usual that these potentials are normalized to be L^2-orthogonal to the constants. Denote also by $\Delta, \tilde{\Delta}$ the respective $\bar{\partial}$-Laplacians of the two metrics. Then we have the following:*

- **A** *If $(\omega, \tilde{\omega})$ form a harmonic pair then $\mathcal{R}^{\tilde{\Omega}}_{\Omega} \equiv 0$.*
- **B** *If $\Delta\Phi$ is a holomorphy potential, and $\mathcal{R}^{\tilde{\Omega}}_{\Omega}(\Xi_{\Delta\Phi}) = 0$, then $(\omega, \tilde{\omega})$ form a harmonic pair.*
- **C** *If $(\omega, \tilde{\omega})$ form a harmonic pair, then g has constant scalar curvature if and only if $\tilde{g}$ has constant scalar curvature.*
- **D** *If $(\omega, \tilde{\omega})$ form a harmonic pair,and g is extremal, then so is $\tilde{g}$ with respect to the same holomorphic vector field if and only if $\Delta^2 F = \tilde{\Delta}^2 F$ and $\Xi_{\Delta F}$ is $\tilde{g}$-gradient.*
- **E** *If both g and $\tilde{g}$ are extremal with respect to the same holomorphic vector field, then $(\omega, \tilde{\omega})$ form a harmonic pair if and only if $\Delta^2 F = \Delta^2 \tilde{F}$.*

For the proof, recall ([2, 17, 10], see also [15]) that a holomorphy potential f satisfies

$$\imath_{\Xi_f}\omega = \bar{\partial} f, \qquad \imath_{\Xi_f}\rho = \bar{\partial}(\Delta f), \tag{5}$$

and on compact manifolds, the latter relation implies the holomorphicity of Ξ_f.

Proof. **A** and **B** are proved as for the Futaki character, with the reflection potential taking the place of the Ricci potential. **C** follows since the assumptions imply $\tilde{\rho}_H = \rho_H = \rho = \tilde{\rho}$. For **D**, the pair being harmonic means $\Phi = 0$, which implies $F = \tilde{F}$. Using (3), ΔF differs by a constant from the scalar curvature of the *extremal* metric g. If $\tilde{g}$ is also extremal with respect to the same vector field, $\Xi_{\Delta F}$ is $\tilde{g}$-gradient, and using twice the second relation in (5),

$$\bar\partial\Delta^2 F = \imath_{\Xi_{\Delta F}}\rho = \imath_{\Xi_{\Delta F}}\tilde\rho = \bar\partial\tilde\Delta^2\tilde F = \bar\partial\tilde\Delta^2 F, \tag{6}$$

so $\Delta^2 F$ and $\tilde\Delta^2 F$ differ by a constant. But now the volume forms of the two metrics are equal, so integration of either of these double Laplacians shows, by the Divergence Theorem, that in fact the constant is zero. Assuming conversely that $\Xi_{\Delta F}$ is $\tilde g$-gradient and $\Delta^2 F = \tilde\Delta^2 F = \tilde\Delta^2\tilde F$, since the first two equalities and the fourth of equation (6) hold, and its first and last terms are equal, the third equality holds as well. Since $\Xi_{\Delta F}$ is $\tilde g$-gradient, the relation $\imath_{\Xi_{\Delta F}}\tilde\omega = \bar\partial\tilde f$ holds for some function $\tilde f$, and since it is holomorphic, $\imath_{\Xi_{\Delta F}}\tilde\rho = \bar\partial\tilde\Delta\tilde f$ follows by (5). Comparing this with the third equality in (6) we see that $\tilde\Delta\tilde F$ is a $\tilde g$-holomorphy potential for $\Xi_{\Delta F}$, since its $\tilde g$-Laplacian equals $\tilde\Delta\tilde f$ up to an additive constant. Hence $\tilde g$ is also extremal with respect to $\Xi_{\Delta F}$. Finally, to show **E**, we note in one direction again that $\Phi = 0$ implies $F = \tilde F$. The assumption in the other direction means that $\Delta(\Delta\tilde F - \Delta F) = 0$, so the difference inside the brackets must be a constant. Integrating with respect to $\omega^{\wedge n}$ and using the Divergence Theorem, again eliminates the constant. Repeating this argument eliminates the remaining Laplacian, and so we see that $F = \tilde F$, $\Phi = 0$, and the pair is harmonic. □

Now **A** alone already gives the first part of the Main Theorem. Its last part is shown in the next section, where we give examples of harmonic Calabi-Yau pairs of extremal metrics.

4. Extremal Harmonic Pairs

In [8], families of extremal Kähler metrics are constructed, generalizing an ansatz given in [16, 11, 12] for the Kähler-Einstein case. We demonstrate the existence of non-trivial harmonic pairs of extremal metrics using only special cases of Hwang's construction. Recall that we are looking for examples which are not product metrics. Instead, the metrics live on projectivized vector bundles over products.

4.1. **The Construction.** The ansatz is given as follows (for details, see the above papers as well as [9]). Let $N := M \times M$ be a product of two copies of a Kähler-Einstein manifold M of positive Ricci curvature, second Betti number equal to one and dimension l. Let ω be an indivisible integral Kähler-Einstein form on M with Ricci form $c_1(M,\omega) = k\omega$, $k > 0$. Let $p : (L,h) \to N$ be a holomorphic Hermitian line bundle having first Chern form $c_1(L,h) = n\omega_1 + n\omega_2$, with h the Hermitian metric, n a positive integer, and $\omega_i = \pi_i^*\omega$ with π_i the corresponding projections on the factors of N. Also, take B to be the symmetric two-tensor associated with $2\pi c_1(L,h)$ via the complex structure on N. Fix positive real numbers $a_1 \neq a_2$ and b, with $a_i \pm bn > 0$, $i = 1, 2$. Let g_N be the Kähler metric on N with Kähler form $a_1\omega_1 + a_2\omega_2$. The Ricci tensor r_N of g_N has constant eigenvalues $k/a_1, k/a_2$, each of multiplicity l with respect to g_N, and B has eigenvalues $n/a_1, n/a_2$, also of multiplicity l.

Define two functions

$$Q(x) := \det(I - x g_N^{-1} B) = \prod_{i=1}^{2}(1 - \frac{n}{a_i}x)^l, \quad T(x) := \mathrm{tr}_{(g_N - xB)}\, r_N = \sum_{i=1}^{2} \frac{kl}{a_i - nx}.$$

Both Q and $T \cdot Q$ are everywhere defined and positive on $(-b, b)$. To emphasize the dependence on the Kähler class (of the base), we will sometimes write Q_{a_1,a_2}, T_{a_1,a_2}. Next use Q and T to define $\phi : [-b, b] \to \mathbb{R}$ by $(\phi Q)(x) = 2(x+b)Q(-b) - 2\int_{-b}^{x}(\sigma_0 + \lambda y - T(y))(x-y)Q(y)\,\mathrm{dy}$, where the constants σ_0 and λ can be written in terms of b, a_i and n, by solving the equations $\sigma_0\alpha_0 + \lambda\alpha_1 = Q(b) + Q(-b) + \int_{-b}^{b} T(x)Q(x)\,\mathrm{dx}$, and $\sigma_0\alpha_1 + \lambda\alpha_2 = b(Q(b) - Q(-b)) + \int_{-b}^{b} xT(x)Q(x)\,\mathrm{dx}$. Here $\alpha_i = \int_{-b}^{b} x^i Q(x)\,\mathrm{dx}$, $i = 0, 1, 2$. Now ϕ is smooth on [-b,b], non-negative, zero exactly at the endpoints, and satisfies $\phi'(\pm b) = \mp 2$. Using ϕ, further define two functions u, t on L_0 by precomposing the hermitian norm with a respective function (denoted by the same letter) having $(0, \infty)$ as domain: $\log r = \int_{-b}^{u(r)}(1/\phi(x))\mathrm{dx}$ and $t(r) = \int_{-b}^{u(r)}(1/\sqrt{\phi(x)})\mathrm{dx}$. Note that u has range $[-b, b]$.

Taking J to be the complex structure on L_0, the above data determine a metric

$$g = dt^2 + (dt \cdot J)^2 + p^* g_N - u p^* B$$

on the complement L_0 of the zero section in L, which extends to an extremal Kähler metric on the compactification $\mathbb{P}(L \oplus \mathbb{C})$ of L_0. Its explicit form allows one to give local coordinate expressions for the various quantities of interest. If z_0 is a fiber coordinate such that $\partial/\partial z_0$ is the generator of the $\mathbb{C}^*$-action, and $z_1, ..., z_{2l}$ are coordinates on N, then on a *fiber where* $\partial u/\partial z_i = 0$, $i = 1, \ldots 2l$, the Ricci tensor is given by $r_{0\bar{0}} = -\phi(\phi' + \frac{Q'}{Q}\phi)'(u)$, $r_{0\bar{\beta}} = 0$, and $r_{\alpha\bar{\beta}} = (r_N)_{\alpha\bar{\beta}} + \frac{1}{2}(\phi(\log(\phi Q))')(u)B_{\alpha\bar{\beta}}$, with the prime denoting differentiation with respect to u. The expression for the scalar curvature is $s(u) = T(u) - (\phi Q)''(u)/(2Q(u))$. Finally, for any smooth function $f : [-b, b] \to \mathbb{R}$,

$$\Delta f(u) = \frac{(\phi Q f')'}{2Q}(u). \tag{7}$$

The condition for such a function to be a holomorphy potential is simply that it be affine in u, a condition that can be verified directly for s from its expression above (in fact, $s(u) = \sigma_0 + \lambda u$). Thus one obtains extremal Kähler metrics in every Kähler class of the Kähler cone of $\mathbb{P}(L \oplus \mathbb{C})$. These will not in general be of constant scalar curvature, since the required extra condition $\lambda = 0$ is obtained (at most) on a real-algebraic hypersurface of the Kähler cone. The latter can be computed explicitly by the defining equations for σ_0 and λ. These metrics all share the same extremal vector field up to a constant multiple, which can be fixed upon normalizing the metrics in their homothety classes.

4.2. **Harmonic Pairs.** We now give isometric pairs of extremal metrics on $\mathbb{P}(L \oplus \mathbb{C})$, related by a diffeomorphism fixing both ρ and ρ_H. The Kähler cone is parameterized by a_1, a_2, b. We normalize by regarding b as fixed, and consider different values of a_1, a_2. To obtain extremal Calabi-Yau pairs from this family, we make the simple observation that $Q_{a_1,a_2} = Q_{a_2,a_1}$, $T_{a_1,a_2} = T_{a_2,a_1}$. Going through the above expressions in succession we see that σ_0, λ, then ϕ, u, t and finally the Ricci tensor, s and Δs all remain invariant under this permutation of the a_i's. So g_{a_1,a_2} and g_{a_2,a_1}, for every allowable value of the a_i's, form an extremal Calabi-Yau pair, in general, of non-constant scalar curvature (hence, in general, the individual Futaki invariants are non-zero). To show that this pair is harmonic, we note that by (3), $s_{a_1,a_2}(u) = s_{a_2,a_1}(u)$ implies $\Delta_{a_1,a_2}F_{a_1,a_2}(u) = \Delta_{a_2,a_1}F_{a_2,a_1}(u)$. Now, since by expression (7) and the above permutation invariances, the Laplacians of the two metrics coincide, as operators on functions of u, the right hand side of the last equality equals $\Delta_{a_1,a_2} F_{a_2,a_1}(u)$. Therefore, so does the left hand side. Operating on the resulting equality again via Δ_{a_1,a_2}, one gets

$$\Delta^2_{a_1,a_2} F_{a_1,a_2}(u) = \Delta^2_{a_1,a_2} F_{a_2,a_1}(u).$$

Thus condition **E** in Proposition 3.3 holds, the pair $(\omega_{a_1,a_2}\,,\,\omega_{a_2,a_1})$ is harmonic, and by condition **A** of the same proposition,

$$\mathcal{R}^{[\omega_{a_2,a_1}]}_{[\omega_{a_1,a_2}]} \equiv 0.$$

References

1. S. Bando, *An obstruction for Chern class forms to be harmonic,* unpublished.
2. S. Bochner, *Vector fields and Ricci curvature,* Bull. Amer. Math. Soc. 52 (1946), 776–797.
3. E. Calabi, *Extremal Kähler metrics,* Seminar on Differential Geometry (S. T. Yau, ed.), Annals of Mathematics Studies 102, Princeton University Press, Princeton, N.J., 1982, 259–290.
4. E. Calabi, *Extremal Kähler metrics II,* Differential Geometry and Complex Analysis (I. Chaval and H.M. Farkas, eds.), Springer, Berlin-New York, 1985, 95–114.
5. A. Futaki, *An obstruction to the existence of Einstein Kähler metrics,* Invent. Math. 73 (1983), 437–443.
6. A. Futaki, *On compact Kähler manifolds of constant scalar curvature,* Proc. Japan Acad., Ser. A, Math. Sci. 59 (1983), 401–402.
7. P. Griffiths and J. Harris, *Principles of Algebraic Geometry,* Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York 1978.
8. A. D. Hwang, *On existence of Kähler metrics with constant scalar curvature,* Osaka J. Math. 31 (1994), 561–595.
9. A. D. Hwang and S. R. Simanca, *Distinguished Kähler metrics on Hirzebruch surfaces,* Trans. Amer. Math. Soc. 347 (1995), 1013–1021.
10. S. Kobayashi, *Transformation Groups in Differential Geometry,* reprint of the 1972 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.
11. N. Koiso and Y. Sakane, *Nonhomogeneous Kähler-Einstein metrics on compact complex manifolds,* Curvature and Topology of Riemannian Manifolds (Katata, 1985), Lecture Notes in Mathematics 1201, Springer, Berlin-New York, (1986), 165–179.
12. N. Koiso and Y. Sakane, *Nonhomogeneous Kähler-Einstein metrics on compact complex manifolds II,* Osaka J. Math. 25 (1988), 933–959.
13. T. Mabuchi, *K-Energy maps integrating Futaki invariants,* Tôhoku Math. J. 38 (1986), 575–593.
14. G. Maschler, *Distinguished Kähler metrics and Equivariant Cohomological Invariants,* thesis, State University of New York at Stony Brook, August 1997.
15. G. Maschler, *Central Kähler metrics,* to be published.
16. Y. Sakane, *Examples of compact Kähler-Einstein manifolds with positive Ricci tensor,* Osaka J. Math. 23 (1986), 585–616.
17. K. Yano, *Sur un théorème de M. Matsushima,* Nagoya Math J. 12 (1957), 147–150.
18. S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I,* Comm. Pure Appl. Math. 31 (1978), 339–411.