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LOCAL CLASSIFICATION OF CONFORMALLY-EINSTEIN KÄHLER METRICS IN HIGHER DIMENSIONS

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Introduction

This is the first of three papers dealing with local and global properties of conformally-Einstein Kähler metrics in higher dimensions; the other two are [**12**], [**13**]. Specifically, we study quadruples (M,g,m,τ) in which

$$(M,g) \text{ is a Kähler manifold of complex dimension } m \text{ and } \tau \text{ is a non-constant } C^\infty \text{ function on } M \text{ such that the conformally related metric } \tilde{g}=g/\tau^2, \text{ defined wherever } \tau\neq 0, \text{ is Einstein.} \tag{0.1}$$

As shown in Proposition 6.4 below, if (0.1) holds with $m\geq 3$, then also

$$M,g,m,\tau \text{ satisfy (0.1) and } d\tau\wedge d\Delta\tau=0 \text{ everywhere in } M, \tag{0.2}$$

so that locally, at points with $d\tau\neq 0$, the Laplacian of τ is a function of τ. As a result, (0.2) is of independent interest only for Kähler *surfaces* ($m=2$).

Our main result (Theorem 24.1) provides a complete local classification, at points in general position, of all M,g,m,τ satisfying (0.1) with $m\geq 3$, or (0.2) with $m=2$. As noted in Remark 24.2, for each fixed m their local biholomorphic-isometry types depend on three real constants plus an additional 'parameter' in the form of an arbitrary local biholomorphic-isometry type of a Kähler-Einstein metric in complex dimension $m-1$.

We derive Theorem 24.1 from a more general local classification (Theorem 18.1) of Kähler manifolds (M,g) with *special Kähler-Ricci potentials*, that is, functions τ satisfying the second-order condition (7.1) in § 7, which also involves the Ricci tensor.

Our model for this latter classification, in any given complex dimension $m\geq 2$, is a connected open set M in the total space of a holomorphic line bundle $\mathcal{L}$ over a Kähler-Einstein manifold (N,h), with a Kähler metric g on M such that the bundle projection $\pi:M\to N$ has totally geodesic fibres, is a *horizontally homothetic submersion* in the sense of [**15**], and shares the horizontal distribution with a $\mathrm{U}(1)$ connection in $\mathcal{L}$, the curvature form of which is proportional to the Kähler form of (N,h). Our choice of the metric g, described in § 8, is in fact even more particular: it depends, essentially, on just one arbitrary C^∞ function $Q>0$ of the special Kähler-Ricci potential $\tau:M\to\mathbf{R}$, treated as an independent variable.

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(See § 8 and Remark 16.1.) Regarded as functions on M, these τ, Q are related by $Q = g(\nabla\tau, \nabla\tau)$. (Many similar constructions are based on Q; cf. [**17, 24**].)

To obtain Theorem 24.1 as a consequence of Theorem 18.1, we observe (in Proposition 22.1 and § 23) that a model M, g, m, τ as above satisfies (0.1) with $m \geq 3$, or (0.2) with $m = 2$, if and only if Q is a specific type of a rational function of τ.

A *global* classification of all M, g, m, τ with (0.1) and $m \geq 3$, or (0.2) and $m = 2$, for which M is compact, can similarly be derived from a global classification of compact Kähler manifolds with special Kähler-Ricci potentials. These classification theorems both require extensive additional arguments, based on two different methods, and will therefore appear in separate papers [**12, 13**].

The simplest examples of quadruples (M, g, m, τ) with (0.1) for which M is compact, $m \geq 2$, and $\tau = 0$ somewhere in M, are provided by some Riemannian products that have apparently been known for decades; see § 25 below. Another family of compact examples with (0.1), representing all dimensions $m > 2$, and this time having $\tau \neq 0$ everywhere in M (so that g is globally conformally Einstein), was constructed by Lionel Bérard Bergery [**2**]; cf. also § 26 below. More recent extensions of the results of [**2**] can be found in [**28**].

Compact Kähler *surfaces* (M, g) with (0.1) have been studied by many authors. For instance, Page [**21**] found the only known example of an Einstein metric on a compact complex surface (namely, on $M = \mathbf{C}\mathrm{P}^2 \# \overline{\mathbf{C}\mathrm{P}^2}$) which is globally conformally-Kähler, but not Kähler; its conformally-related Kähler metric was independently discovered by Calabi ([**6, 7, 9**]. Page's manifold thus satisfies (0.1) (and, in fact, (0.2)) with $m = 2$, and has $\tau \neq 0$ everywhere in M. Examples of (0.2) with $m = 2$ such that M is compact and τ vanishes somewhere are constructed, for minimal ruled surfaces M, in [**16**] and [**27**]. Other results concerning (0.1) for compact Kähler surfaces (M, g) include LeBrun's structure theorem [**20**] for Hermitian Einstein metrics on compact complex surfaces and the variational characterization of such metrics in [**25**]. For non-compact examples, see [**1, 11**].

Condition (0.1) with $m = 2$ is much less restrictive than for $m \geq 3$, as it implies (0.2) in the latter case, but not in the former. This reflects the fact that, at points where the scalar curvature s of a Kähler-surface metric g is non-zero, the metric $\tilde{g} = g/\mathrm{s}^2$ already satisfies a consequence of the Einstein condition (namely, vanishing of the divergence of the self-dual Weyl tensor), and is, up to a constant factor, the only metric conformal to g with this property; see [**10**, top of p. 417].

We wish to thank the referee for suggesting changes that make the present paper easier to read and bringing Gudmundsson's paper [**15**] to our attention.

1. A summary of contents

This paper is organized as follows. Sections 2–5 contain preliminary material. Basic facts on *conformally Einstein metrics* and the definition of *special Kähler-Ricci potentials*, along with their relation to each other and examples of the latter, are presented in §§ 6–9. Sections 10 and 11 deal with some ordinary differential equations associated with the two italicized types of objects. In § 12 we establish 'the ϕ alternative', stating that a certain function ϕ naturally defined by a special Kähler-Ricci potential must be identically zero, or non-zero everywhere. Next, in

§ 13 we show that, essentially, $\phi = 0$ if and only if the underlying Kähler metric is locally reducible. After some preparation in § 14 we introduce in § 15, following [**15**], a simple generalization of Riemannian submersions, needed to verify, in § 16, the claims we make about the examples constructed in § 8, and then, in § 18, to prove our first classification result, Theorem 18.1, which also uses some lemmas from § 17. Theorem 18.1 asserts that locally, at points in general position, any Kähler metric with a special Kähler-Ricci potential looks like one of the examples we describe in § 8.

The remaining sections of the paper deal exclusively with conformally-Einstein Kähler metrics or, more precisely, with quadruples (M, g, m, τ) satisfying (0.1) for $m \geq 3$, or (0.2) for $m = 2$. First, in §§ 19-21, we explicitly solve the corresponding ordinary differential equations of § 10. We use the resulting three distinct families of solutions to introduce, in § 22, three separate types of such quadruples. In § 23 we use them again to describe three families of examples, one for each type, obtained via a special case of the construction in § 8. Then, in § 24 we prove our main classification result, stating that (locally, at points in general position) any quadruple (M, g, m, τ) with (0.1) for $m \geq 3$ or (0.2) for $m = 2$ belongs to one of those three families. Finally, §§ 25-27 are appendices, in which we describe the well-known examples of compact product manifolds with (0.1), explicitly realize Bérard Bergery's metrics [**2**] as a special case of one of our three families, and discuss some geometrically relevant integrals of the differential equations in § 10.

2. Preliminaries

Except for § 3, the symbol ∇ will denote both the Levi-Civita connection of a given Riemannian metric g on a manifold M, and the g-gradient operator acting on C^1 functions $M \to \mathbf{R}$ (see (2.2)). Thus, for C^2 vector fields v, w, u,

$$\text{(2.1)}\qquad \begin{aligned} \text{i)}\quad & 2g(\nabla_w v, u) \,=\, d_w[g(v,u)] \,+\, d_v[g(w,u)] \,-\, d_u[g(w,v)] \\ & \qquad +\, g(v,[u,w]) \,+\, g(u,[w,v]) \,-\, g(w,[v,u])\,, \\ \text{ii)}\quad & \nabla_v w \,-\, \nabla_w v \,=\, [v,w]\,, \qquad \text{that is, } \nabla \text{ is torsion-free} \end{aligned}$$

(see [**18**, Proposition 2.3 on p. 160]), with d_v etc. denoting the directional derivatives. We use the standard notation $\imath_v b = b(v, \cdot, \ldots, \cdot)$ for a multiply-covariant tensor b and a vector (field) v. This will be applied to b which are 1-forms, symmetric twice-covariant tensors (such as the metric g, the Ricci tensor r, or $\nabla d\tau$ with (2.3)), and differential 2-forms. For instance,

$$\text{(2.2)}\qquad \imath_v g \,=\, g(v, \cdot\,) \,=\, d\tau\,, \qquad \text{where} \quad v = \nabla\tau \quad \text{is the gradient of} \quad \tau\,.$$

More generally, let ξ be a 1-form. Then $\imath_v g = \xi$ whenever $\xi = g(v, \cdot\,)$ is the 1-form corresponding to v via g, that is, obtained from v by 'lowering the index'.

The second covariant derivative $\nabla d\tau$ of a C^2 function τ on a Riemannian manifold clearly has the property that, for any vector fields u, w,

$$\text{(2.3)}\qquad (\nabla d\tau)(u,w) \,=\, g(u, \nabla_w v) \,=\, g(\nabla_u v, w)\,, \qquad \text{where} \quad v \,=\, \nabla\tau\,.$$

Remark 2.1. For a C^1 vector field v on a Riemannian manifold (M,g) we will treat the covariant derivative ∇v as a vector-bundle morphism $TM \to TM$ sending each $w \in T_xM$, $x \in M$, to $\nabla_w v \in T_xM$. Thus, by (2.3), for any C^2 function τ on M, the eigenvalues and eigenvectors of the symmetric 2-tensor $\nabla d\tau$ at any point are the same as those of ∇v for $v = \nabla\tau$.

Any C^2 function τ in a Riemannian manifold (M,g) satisfies the relations

$$\text{(2.4)}\quad \begin{array}{rl} \text{i)} & \imath_v d\tau \,=\, d_v\tau \,=\, Q\,, \quad \text{with} \quad v \,=\, \nabla\tau \quad \text{and} \quad Q \,=\, g(\nabla\tau,\nabla\tau)\,, \\ \text{ii)} & 2\imath_v b \,=\, dQ\,, \quad \text{where} \quad v = \nabla\tau, \;\; b = \nabla d\tau \;\; \text{and} \;\; Q = g(\nabla\tau,\nabla\tau)\,, \\ \text{iii)} & \imath_w b \,=\, \nabla_w d\tau \qquad \text{for} \quad b = \nabla d\tau \quad \text{and any vector field} \quad w\,. \end{array}$$

In fact, (2.4.i) is clear as $d_v\tau = g(v,\nabla\tau)$, (2.4.ii) is immediate from the local-coordinate expression $2\tau^{,k}\tau_{,jk} = [\tau^{,k}\tau_{,k}]_{,j}$, and (2.4.iii) is obvious from (2.3).

We denote by $\langle b, b'\rangle$ the natural inner product of twice-covariant tensors b, b' at any point x of a Riemannian manifold (M,g), with

$$\langle b, b'\rangle \,=\, b^{jk}b'_{jk} \,=\, \operatorname{Trace} B^*B'\,, \tag{2.5}$$

where the components $b^{jk} = g^{jp}g^{kq}b_{pq}$ and b'_{jk} refer to any local coordinate system, while $B : T_xM \to T_xM$ is related to b via $b(u,w) = g(Bu,w)$ for all $u, w \in T_xM$ (and similarly for B', b'). In particular, we have the *g-trace* of a twice-covariant tensor b, given by $\operatorname{Trace}_g b = g^{jk}b_{jk}$, and so, for B related to b as above,

$$\operatorname{Trace}_g b \,=\, \langle b, g\rangle \,=\, \operatorname{Trace} B\,. \tag{2.6}$$

For the tensor and exterior products and the exterior derivative of 1-forms ξ, ξ', and C^1 tangent vector fields u, v on any manifold, we have

$$\text{(2.7)}\quad \begin{array}{rl} \text{i)} & (\xi\otimes\xi')(u,v) \,=\, \xi(u)\xi'(v)\,, \qquad\quad \xi\wedge\xi' \,=\, \xi\otimes\xi' \,-\, \xi'\otimes\xi\,, \\ \text{ii)} & (d\xi)(u,v) \,=\, d_u[\xi(v)] \,-\, d_v[\xi(u)] \,-\, \xi([u,v])\,. \end{array}$$

In any Riemannian manifold, the divergence operator δ acts on multiply-covariant C^1 tensor fields b and C^1 vector fields v by $\delta b = \sum_u \imath_u(\nabla_u b)$ and $\delta v = \operatorname{Trace}\nabla v$ (cf. Remark 2.1), with summation over the vectors u of any orthonormal basis of the tangent space at any given point; the two formulae are consistent, as $\delta v = \delta\xi$ whenever $\xi = g(v,\,\cdot\,) = \imath_v g$. Thus, for C^2 functions τ, C^1 vector fields v, w, C^1 1-forms ξ and twice-covariant C^1 tensor fields b,

$$\text{(2.8)}\quad \begin{array}{rl} \text{i)} & \Delta\tau \,=\, \delta(d\tau) \,=\, \delta v\,, \quad \text{where} \;\; v \,=\, \nabla\tau \;\; \text{and} \;\; \Delta \;\; \text{is the Laplacian}, \\ \text{ii)} & \delta[\tau w] \,=\, d_w\tau \,+\, \tau\delta w\,, \quad \text{and} \quad \delta[\tau b] \,=\, \imath_v b \,+\, \tau\delta b \;\; \text{if} \;\; v = \nabla\tau, \\ \text{iii)} & \imath_w[(\imath_v g)\otimes\xi] \,=\, g(v,w)\xi \;\; \text{and} \;\; \delta[(\imath_v g)\otimes\xi] \,=\, (\delta v)\xi \,+\, \nabla_v\xi\,. \end{array}$$

In fact, (2.8.i), (2.8.ii) are obvious, and the two equalities in (2.8.iii) follow from (2.2), (2.7.i) and, respectively, the obvious local-coordinate identity $(v^k\xi_j)_{,k} = v^k{}_{,k}\xi_j + v^k\xi_{j,k}$.

The symbols R, r and s usually stand for the curvature tensor, Ricci tensor and scalar curvature of a given Riemannian metric g, with R as in § 3, $\mathrm{r}(u,w) = \mathrm{Trace}[v \mapsto R(u,v)w]$ for tangent vectors u, v, w, and $\mathrm{s} = \mathrm{Trace}_g \mathrm{r}$ (cf. (2.6)). Thus, with $\imath_w b, \delta b, \delta v$ as above (also for $b = \mathrm{r}$),

$$\text{(2.9)}\quad \begin{array}{ll} \text{a)} & 2\delta\mathrm{r} = d\mathrm{s}\,, \\ \text{b)} & \imath_v\mathrm{r} = \delta b - d\delta v \text{ for any } C^2 \text{ vector field } v, \text{ with } b(u,w) = g(u, \nabla_w v), \\ \text{c)} & \delta b = \imath_v\mathrm{r} + d\Delta\tau \text{ if } v = \nabla\tau \text{ and } b = \nabla d\tau \text{ for any } C^2 \text{ function } \tau. \end{array}$$

The first two equalities, known as the *Bianchi identity for the Ricci tensor* and the *Bochner formula* read, in local coordinates, $2\mathrm{r}_{kj,}{}^k = \mathrm{s}_{,j}$ and $\mathrm{r}_{jk}v^k = v^k{}_{,jk} - v^k{}_{,kj}$ (see, for example, [**3**, Proposition 1.94 on p. 43] and [**3**, formula (2.51)]). The last equality in (2.9.b), with arbitrary tangent vectors u, w, *defines* the twice-covariant tensor field b appearing in the first equality. Finally, (2.9.c) is a special case of (2.9.b) (cf. (2.8.i) and (2.3)).

A C^1 vector field v on a Riemannian manifold (M, g) is locally a gradient if and only if ∇v is self-adjoint at every point (cf. Remark 2.1). This is clear from (2.2), as (2.7.ii) for $\xi = \imath_v g$ gives, by (2.1.ii), $(d\xi)(u,v) = g(\nabla_u v, w) - g(u, \nabla_w v)$.

3. Connections and curvature

In this section we depart from our usual conventions and let ∇ and R stand for an arbitrary (linear) connection in any real/complex vector bundle $\mathcal{E}$ over a manifold and, respectively, the curvature tensor of ∇, with the sign convention $R(u,v)w = \nabla_v\nabla_u w - \nabla_u\nabla_v w + \nabla_{[u,v]}w$ for C^2 vector fields u, v tangent to the base and a C^2 section w of $\mathcal{E}$. Denoting by m the real/complex fibre dimension of $\mathcal{E}$ and choosing C^∞ sections w_a, $a = 1, \ldots, m$, which trivialize the bundle $\mathcal{E}$ over some open subset U of the base, we have, for any vector fields u, v on U,

$$\text{(3.1)}\quad \begin{array}{ll} \text{i)} & \nabla_v w_a = \varGamma_a^c(v) w_c\,, \quad \text{with some real/complex valued 1-forms } \varGamma_a^b, \\ \text{ii)} & R(u,v)w_a = R_a^c(u,v)w_c\,, \quad \text{with} \quad R_a^b = -d\varGamma_a^b + \varGamma_a^c \wedge \varGamma_c^b\,. \end{array}$$

Here we sum over the repeated index $c = 1, \ldots, m$, and $d, \wedge$ are as in (2.7).

Remark 3.1. Given a connection ∇ in a complex line bundle over any manifold, let $\varOmega$ denote the *curvature form* of ∇, that is, the 2-form characterized by $R(u,v)w = i\varOmega(u,v)w$ with $R(u,v)w$ as above. Any local C^∞ section w without zeros then gives rise to the *connection* 1-*form* $\varGamma$ defined by $\nabla_v w = \varGamma(v)w$, that is, (3.1.i) with $m = 1$. Thus, (3.1.ii) with $m = 1$ yields $\varOmega = i\,d\varGamma$.

For any vector bundle $\mathcal{L}$ over a manifold N, we will write

$$\text{(3.2)}\qquad \mathcal{L} = \{(y,z) : y \in N, \ \ z \in \mathcal{L}_y\} \qquad \text{and} \qquad N \subset \mathcal{L}\,,$$

using the same symbol $\mathcal{L}$ for its total space, and identifying N with the zero section formed by all $(y, 0)$ with $y \in N$. We similarly treat each fibre $\mathcal{L}_y$ as a subset of $\mathcal{L}$, identifying it with $\{y\} \times \mathcal{L}_y$. Thus, N and all $\mathcal{L}_y$ are submanifolds of $\mathcal{L}$ with its obvious manifold structure. Being a vector space, every fibre $\mathcal{L}_y$ may also be identified with a subspace of $T_{(y,z)}\mathcal{L}$, for any $z \in \mathcal{L}_y$ (namely, with the tangent space of the submanifold $\{y\} \times \mathcal{L}_y$ at (y,z)). The vertical distribution $\mathcal{V}$

on the total space $\mathcal{L}$ is the subbundle of $T\mathcal{L}$ with the fibres $\mathcal{V}_{(y,z)} = \mathcal{L}_y \subset T_{(y,z)}\mathcal{L}$. In the case where $\mathcal{L}$ is a complex line bundle over N, using the notation of (3.2) and a fixed real number $a \neq 0$, we may define two vertical vector fields (that is, sections of the vertical distribution $\mathcal{V}$) on the total space $\mathcal{L}$ by

$$v(y,z) = az\,, \qquad u(y,z) = iaz\,. \tag{3.3}$$

Remark 3.2. Any Hermitian fibre metric $\langle\,,\rangle$ in a complex line bundle $\mathcal{L}$ over a manifold N is determined by its *norm function* $r : \mathcal{L} \to [0,\infty)$, assigning $|z| = \langle z, z\rangle^{1/2}$ to each $(y,z) \in \mathcal{L}$. As $\mathcal{V}_{(y,z)} = \mathcal{L}_y$ (see above), $\langle\,,\rangle$ may also be treated as a Hermitian fibre metric in the vertical subbundle $\mathcal{V}$ of $T\mathcal{L}$, and then, for v, u given by (3.3), $\langle v,v\rangle = \langle u,u\rangle = a^2 r^2$ and $\mathrm{Re}\,\langle v,u\rangle = 0$, while, on $\mathcal{L} \smallsetminus N$,

$$d_v r = ar \quad \text{and} \quad d_u r = d_w r = 0 \quad \text{for all horizontal vectors } w\,, \tag{3.4}$$

'horizontality' referring to any given connection in $\mathcal{L}$ that makes $\langle\,,\rangle$ parallel.

Let $\varOmega$ and $\mathcal{H}$ be the curvature form (Remark 3.1) and the horizontal distribution of a given connection in a complex line bundle $\mathcal{L}$ over a manifold N. If we use an unnamed local trivializing section of $\mathcal{L}$, defined on an open set $N' \subset N$, to introduce the 1-form $\varGamma$ as in Remark 3.1, and also to identify the portion $\mathcal{L}'$ of $\mathcal{L}$ lying over N' with $N' \times \mathbf{C}$, then, for any $(y,z) \in \mathcal{L}'$ and $(w,\zeta) \in T_{(y,z)}\mathcal{L}'$,

$$(w,\zeta)^{\mathrm{vrt}} = (0, \zeta + \varGamma(w)z)\,, \qquad (w,\zeta)^{\mathrm{hrz}} = (w, -\varGamma(w)z)\,, \tag{3.5}$$

with $...^{\mathrm{vrt}}, ...^{\mathrm{hrz}}$ standing for the $\mathcal{V}$ and $\mathcal{H}$ components relative to the decomposition $T\mathcal{L} = \mathcal{H} \oplus \mathcal{V}$. (In fact, the formula for $(w,\zeta)^{\mathrm{hrz}}$ follows if one writes down the parallel-transport equation using $\varGamma$.) Since $\varOmega = i\,d\varGamma$ (Remark 3.1), we have

$$[\tilde{w}, \tilde{w}']^{\mathrm{vrt}} = i\varOmega(w,w')z \qquad \text{at any} \quad (y,z) \in \mathcal{L} \tag{3.6}$$

(cf. (3.2)), $\tilde{w}, \tilde{w}'$ being the horizontal lifts to $\mathcal{L}$ of any C^∞ vector fields w, w' on N. (This follows from (2.7.ii), as (3.5) gives $\tilde{w} = (w, -\varGamma(w)z)$ at (y,z).) Since $\mathcal{L}_y \subset T_{(y,z)}\mathcal{L}$, both sides of (3.6) may be treated as vectors tangent to $\mathcal{L}$.

4. Kähler metrics and the Ricci form

Except in a few cases (such as Lemma 4.2), we deal with Kähler manifolds in real terms, using real-valued functions and differential forms, real vector fields (rather than complexified ones), and components of tensors in *real* coordinate systems.

For any complex manifold M we denote by J its complex-structure tensor, treated either as a real vector-bundle morphism $J : TM \to TM$ with $J^2 = -1$, or as the multiplication by i in the complex vector bundle TM.

As usual, a Kähler manifold (M,g) is a complex manifold M with a Riemannian metric g that makes the complex-structure tensor J skew-adjoint and parallel. We denote by ω and ρ its Kähler and Ricci forms, so that, if r is the Ricci tensor,

$$\omega(u,v) = g(Ju,v)\,, \quad \rho(u,v) = \mathrm{r}(Ju,v) \quad \text{for } u,v \in T_xM,\ x \in M. \tag{4.1}$$

Skew-symmetry of ω and ρ reflects the fact that g and r are *Hermitian*, in the sense that both $b = g$ and $b = \mathrm{r}$ are symmetric twice-covariant tensors with

$b(Jw,w') = -b(w,Jw')$ for all $w,w' \in T_xM$ and $x \in M$. Thus, $\rho(u,v) = \rho_{jk}u^jv^k$ in any (real) local coordinates, where, by (4.1), $\rho_{jk} = -\,\mathrm{r}_{jl}J_k^l$.

The following well-known lemmas state that ρ is the curvature form (Remark 3.1) of a natural connection in the highest complex exterior power of TM.

Lemma 4.1. *Let u,v be C^2 vector fields on a Kähler manifold (M,g). Then $\mathrm{Trace}_{\mathbf{C}}[R(u,v)] = i\rho(u,v)$, where ρ is the Ricci form and $R(u,v) : TM \to TM$ denotes the complex-linear bundle morphism $w \mapsto R(u,v)w$ defined as in § 3.*

Proof. Let $\langle\,,\rangle$ be the Hermitian fibre metric in TM whose real part is g. Both J and g are ∇-parallel; thus, ∇ is a connection in the *complex* bundle TM, making $\langle\,,\rangle$ parallel, and so $R(u,v)$ is complex-linear, that is, the morphisms $R(u,v)$ and J commute. Since they both are skew-adjoint relative to $\langle\,,\rangle$, their composite F, with fixed vector fields u,v, is self-adjoint. Hence $2i\,\mathrm{Trace}_{\mathbf{C}}[R(u,v)] = 2\,\mathrm{Trace}_{\mathbf{C}}F = \mathrm{Trace}_{\mathbf{R}}F$, the last equality being clear if one evaluates both traces using a complex basis $e_1,\dots,e_m$ of any given tangent space which diagonalizes F and, respectively, the real basis $e_1,\, ie_1,\,\dots\,,e_m,\, ie_m$.

In terms of components relative to any real local-coordinate system, self-adjointness of F (or, skew-adjointness of J) means that $R_{jpkq}J_l^q$ is symmetric in k,l (or, respectively, $\alpha^{jk} = g^{jl}J_l^k$ is skew-symmetric in j,k). The Bianchi identity thus gives $(R_{jpkq} - R_{jkpq} - R_{jqkp})\alpha^{pq} = 0$, that is, $2R_{jpkq}\alpha^{pq} = R_{jkpq}\alpha^{pq} = -R_{jkq}{}^pJ_p^q$, while $R_{jpkq}\alpha^{pq} = g^{pl}R_{jpkq}J_l^q = g^{pl}R_{jplq}J_k^q = -\mathrm{r}_{jq}J_k^q = \rho_{jk}$ (since, as we saw earlier, $\rho_{jk} = -\,\mathrm{r}_{jl}J_k^l$). Combining these equalities and summing against u^jv^k, we get $2\rho(u,v) = -\mathrm{Trace}_{\mathbf{R}}F = -2i\,\mathrm{Trace}_{\mathbf{C}}[R(u,v)]$, which completes the proof. ∎

Lemma 4.2. *Let ρ be the Ricci form of a Kähler manifold (M,g) of complex dimension m with C^∞ vector fields w_a, $a = 1,\dots,m$, on an open set $U \subset M$ which trivialize the tangent bundle TU treated as a complex vector bundle, and let Γ_a^b be the complex-valued 1-forms with* (3.1.i) *for the Levi-Civita connection ∇ of g. Then $\rho = i\,d\Gamma_a^a$ on U, with summation over $a = 1,\dots,m$.*

In fact, the matrix $[R_a^b(u,v)]$ (see (3.1.ii)) represents the operator $R(u,v)$ of Lemma 4.1 in the complex basis $w_a(x)$ at any point $x \in U$. Now $i\rho = R_a^a = -d\Gamma_a^a$ by Lemma 4.1 and (3.1.ii), with $\Gamma_a^c \wedge \Gamma_c^a = 0$ due to obvious cancellations. ∎

5. Killing potentials

A real-valued C^∞ function τ on a Kähler manifold (M,g) is called a *Killing potential* if $u = J(\nabla\tau)$ is a Killing vector field on (M,g).

Remark 5.1. As usual, a *Killing field* on a Riemannian manifold (M,g) is any C^∞ vector field u such that ∇u is skew-adjoint at every point. Then u is uniquely determined by $u(x)$ and $(\nabla u)(x)$ at any given point $x \in M$. In fact, u is a Jacobi field along any geodesic, since its local flow, applied to a geodesic segment, generates a variation of geodesics. This implies a *unique continuation property*: a Killing field is uniquely determined by its restriction to any non-empty open set.

We call a (real) C^∞ vector field v on a complex manifold *holomorphic* if $\mathbf{L}_vJ = 0$, where $\mathbf{L}$ is the Lie derivative. For a C^∞ vector field v on a *Kähler* manifold,

$$\text{(5.1)}\quad \begin{array}{ll} \text{a)} & v \text{ is holomorphic if and only if } [J,\nabla v]=0,\\ \text{b)} & [v,u]=0 \text{ and } u \text{ is holomorphic if } v \text{ is holomorphic and } u=Jv,\\ \text{c)} & \nabla u = J\circ(\nabla v), \text{ with the convention of Remark 2.1, if } u=Jv. \end{array}$$

In fact, (5.1.a) (or, $[v,u]=0$ in (5.1.b)) follows if we apply the relation $(\mathbf{L}_vJ)w = \mathbf{L}_v(Jw)-J(\mathbf{L}_vw)$, valid for any C^2 vector fields v,w, to any w, obtaining $\mathbf{L}_vJ = [J,\nabla v]$ due to (2.1.ii) with $\nabla J=0$ (or, to $w=v$, noting that $\mathbf{L}_vw=[v,w]$). Now (5.1.c) is clear as $\nabla_wu = J(\nabla_wv)$ for every tangent vector w (since $\nabla J=0$), while (5.1.a) and (5.1.c) yield the remainder of (5.1.b).

Lemma 5.2. *For a C^∞ function τ on a Kähler manifold (M,g), the following three conditions are equivalent*: (i) *τ is a Killing potential*; (ii) *the gradient $v=\nabla\tau$ is a holomorphic vector field*; (iii) *$b=\nabla d\tau$ is Hermitian, as in § 4.*

In fact, let $v=\nabla\tau$ and $u=Jv$. By (5.1.c), $\nabla u = J\circ(\nabla v)$ and $[\nabla v]^*=\nabla v$, that is, ∇v is self-adjoint at every point (cf. Remark 2.1). Thus, $\nabla u+[\nabla u]^* = J\circ(\nabla v)+[\nabla v]^*\circ J^* = [J,\,\nabla v]$ and, by (5.1.a), (i) is equivalent to (ii). As g is Hermitian, (2.3) gives $(\nabla d\tau)(Jw',w) = g(Jw',\nabla_wv) = -g(w',\nabla_wu)$, so that Hermitian symmetry of $\nabla d\tau$ amounts to skew-adjointness of ∇u, that is, to (i). ■

Next, there is a well-known local one-to-one correspondence between Killing potentials defined up to an additive constant, and holomorphic Killing vector fields:

Lemma 5.3. *Let (M,g) be a Kähler manifold. For every Killing potential τ on (M,g), the Killing field $J(\nabla\tau)$ is holomorphic. Conversely, if $H^1(M,\mathbf{R})=\{0\}$, then every holomorphic Killing vector field on (M,g) has the form $J(\nabla\tau)$ for a Killing potential τ, which is unique up to an additive constant.*

The first assertion is clear from Lemma 5.2(ii) and (5.1.b). Next, setting $v=-Ju$ for a holomorphic Killing field u, we get $\nabla v = -J\circ(\nabla u)$ (by (5.1.c)). As J and ∇u commute (cf. (5.1.a)) and are skew-adjoint, their composite ∇v is self-adjoint, that is, locally, $v=\nabla\tau$ for some function τ (see the end of § 2). ■

Remark 5.4. Let τ be a non-constant Killing potential on a Kähler manifold (M,g). Then $\nabla d\tau\neq 0$ wherever $d\tau=0$ (and hence $d\tau\neq 0$ on a dense open subset of M, which also follows from Lemma 5.2(ii)). In fact, if $\nabla d\tau$ and $d\tau$ both vanished at some point, so would $v=\nabla\tau$ and ∇v (by (2.3)), as well as $u=Jv$ and ∇u (since $\nabla u = J\circ\nabla v$ by (5.1.c)). The Killing field $u=J(\nabla\tau)$ thus would vanish identically on M (see Remark 5.1), contradicting the non-constancy of τ.

Lemma 5.5. *Let $\tau : M\to\mathbf{R}$ be a Killing potential on a Kähler manifold (M,g). If we set $v=\nabla\tau$, $u=Jv=J(\nabla\tau)$, $b=\nabla d\tau$ and*

$$\beta(w,w') = b(Jw,w') \qquad \text{for all} \quad w,w'\in T_xM \quad \text{and} \quad x\in M\,, \tag{5.2}$$

then β is a differential 2-form on M, and

$$2\beta = d\xi\,, \qquad \text{where} \quad \xi = \imath_v\omega = \imath_ug\,. \tag{5.3}$$

In fact, $\imath_v\omega=\imath_ug$ by (4.1) (cf. § 2), while Lemma 5.2(iii) implies skew-symmetry of $\beta(w,w')$ in w,w'. Also, (2.7.ii) gives $(d\xi)(w,w') = g(Aw,w')$ with

$A = \nabla u - [\nabla u]^*$, for any C^1 vector fields u, w, w' and $\xi = \imath_u g = g(u,\,\cdot\,)$ (cf. Remark 2.1). As τ is a Killing potential, $u = Jv$, with $v = \nabla\tau$, is a Killing field, so that $A = 2\nabla u$ and $(d\xi)(w, w') = 2g(\nabla_w u, w') = 2g(J\nabla_w v, w') = -2g(\nabla_w v, Jw') = -2b(w, Jw') = 2\beta(w, w')$ in view of (2.3) with $b = \nabla d\tau$ and (5.2). ∎

For any Killing potential τ on a Kähler manifold (M, g), we have (see [**6**])

$$2\imath_v \mathrm{r} \;=\; -dY\,, \qquad \text{where} \quad v = \nabla\tau \quad \text{and} \quad Y = \delta v = \Delta\tau, \tag{5.4}$$

r being the Ricci tensor. In fact, $J_k^p u^k = -v^p$ (that is, $Ju = -v$) for the Killing field $u = Jv$, while $J_l^p u_p = \tau_{,l}$ since $w^l J_l^p u_p = g(Jw, u) = -g(w, Ju) = g(v, w) = d_w\tau = w^l\tau_{,l}$ for any vector w. Relations $\nabla J = 0$ and $u_{p,k} + u_{k,p} = 0$ (see Remark 5.1) yield $u^k{}_{,k} = \delta u = 0$ and $\tau_{,lk}{}^k = (J_l^p u_p)_{,k}{}^k = J_l^p u_{p,k}{}^k = -J_l^p u_{k,p}{}^k$. This in turn equals $-J_l^p \mathrm{r}_{pk} u^k = \rho_{kl} u^k = -\rho_{lk} u^k = \mathrm{r}_{lp} J_k^p u^k = -\mathrm{r}_{lp} v^p$, since $\rho_{jk} = -\mathrm{r}_{jl} J_k^l$ (see § 4) and the coordinate form of identity (2.9.b), with $u^k{}_{,k} = 0$, gives $\mathrm{r}_{jk} u^k = u^k{}_{,jk}$. Therefore, $\delta b = -\imath_v \mathrm{r}$, with v, b as in (2.9.c), which, by (2.9.c), implies (5.4).

6. Conformally-Einstein Kähler metrics

Let $\mathrm{r}, \tilde{\mathrm{r}}$ and $\mathrm{s}, \tilde{\mathrm{s}}$, be the Ricci tensors and scalar curvatures of conformally related Riemannian metrics g and $\tilde g = g/\tau^2$ in real dimension n. Then (see, for example, [**10**, p. 411]), with ∇ and $\Delta = g^{jk}\nabla_j\nabla_k$ denoting the g-gradient and g-Laplacian,

$$\begin{aligned} \tilde{\mathrm{r}} \;&=\; \mathrm{r} \;+\; (n-2)\,\tau^{-1}\nabla d\tau \;+\; \left[\tau^{-1}\Delta\tau \;-\; (n-1)\tau^{-2}Q\right] g\,, \\ \tilde{\mathrm{s}} \;&=\; \mathrm{s}\tau^2 \;+\; 2(n-1)\,\tau\Delta\tau \;-\; n(n-1)Q\,, \qquad \text{where} \quad Q = g(\nabla\tau, \nabla\tau). \end{aligned} \tag{6.1}$$

Thus, for $n \ge 3$, the metric $\tilde g = g/\tau^2$ is Einstein if and only if

$$\nabla d\tau \;+\; (n-2)^{-1}\tau\mathrm{r} \;=\; \sigma g\,, \qquad \text{with} \quad n\sigma \;=\; \Delta\tau \;+\; (n-2)^{-1}\mathrm{s}\tau\,. \tag{6.2}$$

Lemma 6.1. *Let τ be a C^∞ function on a Kähler manifold (M, g) such that, denoting by* r *the Ricci tensor, we have*

$$\nabla d\tau \;+\; \chi\,\mathrm{r} \;=\; \sigma g \qquad \text{for some} \quad C^\infty \quad \text{functions} \quad \chi, \sigma\,. \tag{6.3}$$

Then τ is a Killing potential, that is, $u = J(\nabla\tau)$ is a Killing field on (M, g), and

$$\begin{aligned} &\text{i)} \quad d\xi \;=\; 2\,[\sigma\omega \;-\; \chi\rho]\,, \qquad \text{where} \quad \xi = \imath_v\omega \quad \text{and} \quad v = \nabla\tau\,, \\ &\text{ii)} \quad d\sigma\wedge\omega \;=\; d\chi\wedge\rho\,. \end{aligned} \tag{6.4}$$

In fact, τ is a Killing potential in view of Lemma 5.2(iii), as Hermitian symmetry of $\nabla d\tau$ follows from that of g and r via (6.3) (cf. (4.1)); now (6.4.i) is clear from (5.3) since, for $b = \nabla d\tau$, the 2-form β with (5.2) equals $\sigma\omega - \chi\rho$ (by (6.3) and (4.1)); and, applying d to (6.4.i), we obtain (6.4.ii), as $d\omega = d\rho = 0$. ∎

According to Lemma 6.1, condition (6.3) imposed on a C^∞ function τ on a Kähler manifold (M, g) guarantees that τ is a Killing potential. A few special cases of (6.3) are of independent interest and have been studied extensively. First, (6.3) holds if τ is a constant function on a Riemannian manifold (M,g) which is Einstein or 2-dimensional; more generally, for any g,τ for which $\tilde{g} = g/\tau^2$ is Einstein, we have (6.3) in the form of (6.2); next, Kähler metrics on compact manifolds with functions τ satisfying (6.3) for *constants* χ,σ such that $\chi\sigma > 0$ are known as *Kähler-Ricci solitons* (see [**22, 26, 8**]); finally, Riemannian (or pseudo-Riemannian) manifolds (M,g) admitting non-constant C^∞ functions τ that satisfy (6.3) with $\chi = 0$ have been studied extensively, and their local structure is completely understood (cf. [**19**] and the references therein). See also the comment following (7.1) in § 7.

Remark 6.2. Discussing conformally-Einstein metrics, we always assume, as in (0.1), that τ is a C^∞ function on a given Riemannian manifold (M,g) such that the metric $\tilde{g} = g/\tau^2$, *defined wherever* $\tau \neq 0$, is Einstein. Thus, τ may still vanish somewhere in M. Although relation (6.2) and its consequences, Lemma 6.3 and (6.5) below, are directly established only in the open set where $\tau \neq 0$, they are automatically valid *at every point of* M. In fact, the equalities in question hold both in the set where $\tau \neq 0$, and in the interior of the preimage $\tau^{-1}(0)$, while the union of these two open sets is clearly dense in M.

Lemma 6.3. *Let* M,g,m,τ *satisfy* (0.1) *with* $m \geq 2$. *Then* τ *is a Killing potential on* (M,g)*, that is,* $u = J(\nabla\tau)$ *is a Killing field.*

This is clear from Lemma 6.1, since (6.2) is a special case of (6.3). ■

Assuming (0.1) with $m \geq 2$ and setting $Q = g(\nabla\tau,\nabla\tau)$, $Y = \Delta\tau$, we have

$$\text{(6.5)}\qquad \begin{array}{ll} \text{a)} & (2m-1)(m-2)\,dY + (m-1)\tau d\mathrm{s} - \mathrm{s}\,d\tau = 0, \\ \text{b)} & \tau^2 d\mathrm{s} + 2Y d\tau + 2(m-1)\tau dY - 2m dQ = 0. \end{array}$$

In fact, applying $\imath_v$ for $v = \nabla\tau$ (or, d) to (6.2) (or, to $\tilde{\mathrm{s}}$ in (6.1)), with $n = 2m$, and using (2.4.ii), (5.4) and (2.2) (or, constancy of $\tilde{\mathrm{s}}$, which is the scalar curvature the Einstein metric $\tilde{g}$), we obtain $\mathrm{s}\tau d\tau + 2(m-1)Y d\tau + m\tau dY - 2m(m-1)dQ = 0$ or, respectively, $2\mathrm{s}\tau d\tau + \tau^2 d\mathrm{s} + 2(2m-1)[Y d\tau + \tau dY] - 2m(2m-1)dQ = 0$. Adding $1-2m$ times the former equality to $m-1$ times the latter (or, subtracting twice the former from the latter), we now get (6.5.a) (or, (6.5.b)). (To obtain (6.5.a), we cancelled a τ factor; this is allowed as (6.5.a) still holds in the interior of the preimage $\tau^{-1}(0)$, cf. Remark 6.2.)

Proposition 6.4. *Suppose that* M,g,m,τ *satisfy* (0.1) *with* $m \geq 3$. *Then, locally in the open set where* $d\tau \neq 0$*, each of the three functions* s, $g(\nabla\tau,\nabla\tau)$, $\Delta\tau$ *is a function of* τ. *Here* s *stands, as usual, for the scalar curvature of* g.

In fact, d applied to (6.5.a) (or, (6.5.b)) gives $d\tau \wedge d\mathrm{s} = 0$ (or, $d\tau \wedge dY = 0$, as $d\tau \wedge d\mathrm{s} = 0$). Hence, applying $d\tau \wedge \ldots$ to (6.5.b) we get $d\tau \wedge dQ = 0$. ■

A weaker assertion for $m = 2$ is well-known (see [**10**, Proposition 3 on p. 416]):

Proposition 6.5. *Let* M, g, m, τ *satisfy* (0.1) *with* $m = 2$. *Then the scalar curvature* s *is a constant multiple of* τ. *Thus, either* s *is identically zero, or* τ *is a constant multiple of* s.

This is clear as (6.5.a) with $m = 2$ gives $\tau d\mathrm{s} - \mathrm{s}\, d\tau = 0$, so that $\tau^{-1}\mathrm{s}$ is constant on each connected component U' of the open set $U \subset M$ on which $\tau \neq 0$. Thus, in view of Lemma 6.3 and denseness of U in M (immediate from Remark 5.4), $J(\nabla\tau)$ and $J(\nabla\mathrm{s})$ are Killing fields; due to their unique continuation property (Remark 5.1), the proportionality relation $J(\nabla\mathrm{s}) = pJ(\nabla\tau)$ with $p = \tau^{-1}\mathrm{s}$, valid on U', must hold everywhere in M. Hence $\tau^{-1}\mathrm{s}$ is constant on U. ■

Using well-known analogues of (6.1) for δW (the divergence of the Weyl tensor of a given metric g) and $\tilde{\delta}\tilde{W}$ (its counterpart for $\tilde{g} = g/\tau^2$), one verifies that, for a Kähler metric g in complex dimension m, with the Kähler form ω, we have $(\tilde{\delta}\tilde{W})\omega = 0$ if and only if $2(2m-1)(m-2)\imath_v\mathrm{r} = (m-1)\tau d\mathrm{s} - \mathrm{s}\, d\tau$. This is a generalization of equation (6.5.a): if $\tilde{g}$ is Einstein, $\tilde{\delta}\tilde{W}$ must vanish, while, by (5.4), $2(2m-1)(m-2)\imath_v\mathrm{r} = -(2m-1)(m-2)\, dY$.

7. Special Kähler-Ricci potentials

We call τ a *special Kähler-Ricci potential* on a Kähler manifold (M, g) if

(7.1) τ is a non-constant Killing potential on (M, g) (see § 5) such that, at every point with $d\tau \neq 0$, all non-zero tangent vectors orthogonal to $v = \nabla\tau$ and $u = Jv$ are eigenvectors of both $\nabla d\tau$ and the Ricci tensor r.

Examples are described in § 8 and Corollary 9.3 below. Moreover, (7.1) is closely related to (6.3): assuming (7.1) we get (6.3) on a suitable open set (Remark 7.4), while (6.3) plus some strong additional conditions gives (7.1) (Corollary 9.2).

It is because of this relation with (6.3) that we use the phrase 'Kähler-Ricci potential', adding the word 'special' to indicate an extra assumption (the 'eigenvector clause') made in (7.1). For more on special Kähler-Ricci potentials, see [**12**].

Remark 7.1. For a distribution $\mathcal{V}$ on a Riemannian manifold (M, g) and a symmetric twice-covariant tensor b at a point $x \in M$, consider the following conditions

(7.2)
a) all non-zero vectors in $\mathcal{V}_x$ and $\mathcal{H}_x = \mathcal{V}_x^\perp$ are eigenvectors of b;
b) all non-zero vectors in $\mathcal{H}_x = \mathcal{V}_x^\perp$ are eigenvectors of b.

(Note that (7.2.b) gives $b = \lambda g$ on $\mathcal{H}_x$ for some $\lambda \in \mathbf{R}$.) Let $\mathcal{V}$ now be a J*-invariant* distribution of complex dimension one on a Kähler manifold (M, g). For those symmetric twice-covariant tensors b at a point $x \in M$ which are also Hermitian (see § 4), condition (7.2.b) then implies (7.2.a). In fact, the operator $B : T_xM \to T_xM$ with $b(w, w') = g(Bw, w')$ for all $w, w' \in T_xM$ is self-adjoint, commutes with J, and $B\mathcal{V}_x^\perp \subset \mathcal{V}_x^\perp$. Hence $B\mathcal{V}_x \subset \mathcal{V}_x$. Choosing $v \in \mathcal{V}_x \smallsetminus \{0\}$ and $\mu \in \mathbf{R}$ with $Bv = \mu v$, we thus have $Bu = \mu u$ for $u = Jv$ (as $BJv = JBv$), which yields (7.2.a) since $\dim_{\mathbf{R}}\mathcal{V}_x = 2$.

Definition 7.2. Given a special Kähler-Ricci potential $\tau : M \to \mathbf{R}$ on a Kähler manifold (M, g), as in (7.1), let $M' \subset M$ be the open set on which $d\tau \neq 0$, and let the vector fields v, u on M and distributions $\mathcal{H}, \mathcal{V}$ on M' be given by

$$(7.3) \qquad \mathcal{V} = \operatorname{Span}\{v, u\} \quad \text{and} \quad \mathcal{H} = \mathcal{V}^\perp, \quad \text{with} \quad v = \nabla\tau \quad \text{and} \quad u = Jv\,.$$

The *septuple* $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ *of functions on* M' *associated with* τ consists of $Q = g(\nabla\tau, \nabla\tau)$, $Y = \Delta\tau$, the scalar curvature s, and the 'eigenvalue functions' ϕ, ψ, λ, μ with $\phi = \lambda = 0$ when $\dim_{\mathbf{C}} M = 1$, and, in general,

$$(7.4) \qquad \begin{array}{ll} \mathrm{r} = \lambda g \quad \text{on} \quad \mathcal{H}, & \mathrm{r} = \mu g \quad \text{on} \quad \mathcal{V}, \\ \nabla d\tau = \phi g \quad \text{on} \quad \mathcal{H}, & \nabla d\tau = \psi g \quad \text{on} \quad \mathcal{V}, \\ \mathrm{r}(\mathcal{H},\mathcal{V}) = (\nabla d\tau)(\mathcal{H},\mathcal{V}) = \{0\} & \text{for} \quad \mathcal{H},\mathcal{V} \quad \text{as in} \quad (7.3). \end{array}$$

The last line states that $\mathcal{H}, \mathcal{V}$ are r-orthogonal and $\nabla d\tau$-orthogonal to each other (cf. the eigenvector clause of (7.1)). The rest of (7.4) now follows, for some C^∞ functions λ, μ, ϕ, ψ on M', as (7.2.b) gives (7.2.a) (Remark 7.1). Also, by (7.3),

$$(7.5) \qquad g(v,v) = g(u,u) = Q\,, \qquad g(v,u) = 0 \qquad \text{everywhere in} \quad M.$$

Remark 7.3. Conversely, (7.1) obviously holds for any non-constant Killing potential τ on a Kähler manifold which satisfies (7.4) on the open set where $d\tau \neq 0$, with $\mathcal{V}, \mathcal{H}$ given by (7.3) and some ϕ, ψ, λ, μ.

Remark 7.4. Let τ be a C^∞ function on a Riemannian manifold (M, g) such that some distribution $\mathcal{V}$ on M satisfies (7.2.a) at every $x \in M$, both for $b = \nabla d\tau$ and the Ricci tensor $b = \mathrm{r}$. Then (6.3) holds in the open set of all points at which r is not a multiple of g (that is, $\mathrm{r} \neq \mathrm{s}g/n$, with $n = \dim_{\mathbf{R}} M$).

In fact, the space of all symmetric twice-covariant tensors b with (7.2.a) at a fixed x is two-dimensional, with the eigenvalues serving as parameters. Hence $\nabla d\tau$ is a combination of r and g at points where they are linearly independent.

Thus, every special Kähler-Ricci potential τ on a Kähler manifold satisfies (6.3) on the set where $\mathrm{r} \neq \mathrm{s}g/n$ and $d\tau \neq 0$ (by (7.1), as (7.4) then implies (7.2.a)).

Lemma 7.5. *Let* $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ *be the septuple associated, as above, with a special Kähler-Ricci potential* τ *on a Kähler manifold* (M, g) *of complex dimension* $m \geq 1$. *Then, in the open set* M' *where* $d\tau \neq 0$, *we have* $dQ = 2\psi\, d\tau$, $dY = -2\mu\, d\tau$ *and* $(m-2)\nabla\lambda = -[\nabla\lambda]^{\mathrm{vrt}} + 2(m-1)(\mu - \lambda)\phi v/Q$, *where* $v = \nabla\tau$ *and* $...^{\mathrm{vrt}}$ *stands for the* $\mathcal{V}$ *component relative to the decomposition* $TM' = \mathcal{H} \oplus \mathcal{V}$; *cf.* (7.3). *Furthermore, if* $m \neq 2$ *then* $Q\, d\lambda = 2(\mu - \lambda)\phi\, d\tau$, *while, for any* $m \geq 1$,

(i) $\mathrm{s} = 2\mu + 2(m-1)\lambda$ *and* $Y = 2\psi + 2(m-1)\phi$,
(ii) $\nabla_v v = -\nabla_u u = \psi v$ *and* $\nabla_v u = \nabla_u v = \psi u$, *with* $u = Jv$.

Proof. As $\mathrm{s} = \mathrm{Trace}_g \mathrm{r}$ and $Y = \Delta\tau = \mathrm{Trace}_g(\nabla d\tau)$ (cf. (2.6)), (7.4) proves (i). Since v is a section of $\mathcal{V}$, (7.4) and (2.2) give $\imath_v \mathrm{r} = \mu\, \imath_v g = \mu\, d\tau$ and $\imath_v(\nabla d\tau) = \psi\, d\tau$ (cf. § 2). Now, by (5.4) and (2.4.ii), $dY = -2\mu\, d\tau$ and $dQ = 2\psi\, d\tau$.

By (7.4) and Remark 2.1, $\nabla_v v = \psi v$ and $\nabla_u v = \psi u$, As $u = Jv$ and $\nabla J = 0$, this gives $\nabla_u u = -\psi v$. Now (ii) follows since (5.1.b) and (2.1.ii) yield $\nabla_v u = \nabla_u v$.

Setting $b = d\tau \otimes d\tau + \xi \otimes \xi$ on M', with $\xi = \imath_u g$, we have $\delta b = Y d\tau$, by (2.8.iii) with $\imath_v g = d\tau$ (see (2.2)), or $\xi = \imath_u g$, as $\delta v = Y$ by (2.8.i), while $\delta u = 0$ since u is a Killing field (cf. Remark 5.1) and, finally, $\nabla_v(d\tau) = -\nabla_u\xi = \psi\, d\tau$ (due to the first relation in (ii), with v, u replaced by the corresponding 1-forms $d\tau, \xi$). Also, by (2.8.iii) and (7.5), $\imath_v b = Q\, d\tau = Q\imath_v g$, $\imath_u b = Q\xi = Q\imath_u g$, and $\imath_w b = 0$ for vectors w orthogonal to both v and u. Thus, $\mathrm{r} - \lambda g = b'$, with $b' = (\mu - \lambda)b/Q$ since, by (7.4), both sides have the same image under $\imath_w$ for any tangent vector w. Combined with (i), (2.8.ii) and the relations $\delta b = Y d\tau$ and $dQ = 2\psi\, d\tau$, that is, $\nabla Q = 2\psi v$, the above expressions for $\imath_v b$, $\imath_u b$ and $\imath_w b$ also show that $\delta b' = \imath_w g$ for the vector field $w = [\nabla\mu]^{\mathrm{vrt}} - [\nabla\lambda]^{\mathrm{vrt}} + 2(m-1)(\mu-\lambda)\phi v/Q$. However, $\imath_w g = \delta b' = \delta(\mathrm{r} - \lambda g)$ (since $b' = \mathrm{r} - \lambda g$) which, as $2\delta(\mathrm{r} - \lambda g) = d\mathrm{s} - 2d\lambda$ (by (2.9.a), (2.8.ii) and (2.2)), gives $\imath_w g = d\mu + (m-2)\, d\lambda$ (by (i)), and so $w = \nabla\mu + (m-2)\nabla\lambda$ (cf. (2.2)).

As $dY = -2\mu\, d\tau$, we have $0 = ddY = -2\, d\mu \wedge d\tau$, so that $\nabla\mu$ equals a function times $\nabla\tau$ and, in particular, $[\nabla\mu]^{\mathrm{vrt}} = \nabla\mu$. Comparing the above two expressions for w, we thus obtain the required formula for $(m-2)\nabla\lambda$. When $m > 2$, this shows that $\nabla\lambda$ is a section of $\mathcal{V} = \mathrm{Span}\,\{v, u\}$, that is, $[\nabla\lambda]^{\mathrm{vrt}} = \nabla\lambda$, and hence $\nabla\lambda = 2(\mu - \lambda)\phi v/Q$, that is, $Q\, d\lambda = 2(\mu - \lambda)\phi\, d\tau$. Since both sides of the last equality vanish when $m = 1$ (Definition 7.2), this completes the proof. ■

Note that, by Lemma 7.5(ii), the distribution $\mathcal{V}$ on M' is integrable and has totally geodesic leaves.

8. Examples

To describe examples of special Kähler-Ricci potentials τ on Kähler manifolds (M', g) (see (7.1)), we assume that the following data are given:

$$\mathcal{J}, r, \theta, \tau, f; \qquad a, \varepsilon, c; \qquad m, N, h; \qquad \mathcal{L}, \pi, \mathcal{V}; \qquad \mathcal{H}, \langle\,,\rangle, M'. \tag{8.1}$$

Here $\mathcal{J} \subset (0, \infty)$ is an open interval, $r \in \mathcal{J}$ is a real variable, $a \in \mathbf{R} \smallsetminus \{0\}$ and $\varepsilon \in \{-1, 0, 1\}$ are constants, while $\theta, \tau, f : \mathcal{J} \to \mathbf{R}$ are C^∞ functions such that $\theta > 0$, $f > 0$, $d\tau/dr = ar\theta$, and either $\varepsilon = 0$ and $f = 1$, or $\varepsilon = \pm 1$ and $f = 2\varepsilon(\tau - c)$ with a constant c. Furthermore, $m \geq 1$ is an integer and (N, h) is a Kähler manifold of complex dimension $m - 1$, which we assume to be Einstein unless $m = 2$. Also, $\mathcal{L}$ is a holomorphic line bundle with a Hermitian fibre metric $\langle\,,\rangle$ over N, while π and $\mathcal{V}$ denote the bundle projection $\mathcal{L} \to N$ and the vertical distribution in $\mathcal{L}$ (§ 3). Next, $\mathcal{H}$ is the horizontal distribution of a linear connection in $\mathcal{L}$ making $\langle\,,\rangle$ parallel, whose curvature form Ω (Remark 3.1) equals $-2\varepsilon a\, \omega^{(h)}$, where $\omega^{(h)}$ is the Kähler form of (N, h). We also assume that $\mathcal{H}$ is J-invariant as a subbundle of $T\mathcal{L}$, where $J : T\mathcal{L} \to T\mathcal{L}$ denotes the complex structure tensor of $\mathcal{L}$. (Cf. Remark 17.4.) Finally, M' is a connected open subset of $\mathcal{L} \smallsetminus N$ contained in $r^{-1}(\mathcal{J})$ (the r-preimage of $\mathcal{J}$), where the symbol r stands for the norm function of $\langle\,,\rangle$ as well (see Remark 3.2), so that f, θ, τ and other C^∞ functions of $r \in \mathcal{J}$ may be treated as C^∞ functions $M' \to \mathbf{R}$.

A metric g on M' now is defined by requiring that $g(\mathcal{H}, \mathcal{V}) = \{0\}$, that is, $\mathcal{H}$ be g-orthogonal to $\mathcal{V}$, while $g = f\pi^*h$ on $\mathcal{H}$ and $g = \theta\,\mathrm{Re}\langle\,,\rangle$ on $\mathcal{V}$, where $\mathrm{Re}\langle\,,\rangle$ is the standard Euclidean metric on each fibre of $\mathcal{L}$.

Our M', being an open submanifold of $\mathcal{L}$, is a complex manifold of complex dimension $m \geq 1$. We will verify later, in § 16, that g is a Kähler metric on M', and τ is a special Kähler-Ricci potential on (M', g), as defined in (7.1).

9. More on the conformally-Einstein case

Lemma 9.1. *Let a non-constant C^∞ function $\tau : M \to \mathbf{R}$ on a Kähler manifold (M, g) of any complex dimension $m \geq 2$ satisfy* (6.3) *with $\chi, \sigma : M \to \mathbf{R}$ such that $d\sigma \wedge d\tau = d\chi \wedge d\tau = 0$, that is, any point with $d\tau \neq 0$ has a neighborhood on which both χ, σ are functions of τ. Also, let $\mathcal{V}, \mathcal{H}$ be given by* (7.3) *on the open set $M' \subset M$ where $d\tau \neq 0$. Then, in the open subset of M' on which $d\chi \neq 0$, we have* (7.4) *for some C^∞ functions λ, μ, ϕ, ψ with $d\sigma = \lambda\, d\chi$.*

Proof. Let $b = (d\chi/d\tau)\,\mathrm{r} - (d\sigma/d\tau) g$ on M'. (Note that b is well-defined: $d\chi/d\tau$ and $d\sigma/d\tau$ make sense as $d\sigma \wedge d\tau = d\chi \wedge d\tau = 0$.) Then, with $\imath_w$ as in § 2, $\mathcal{H}_x \subset \{w \in T_xM : \imath_w b = 0\}$ at every $x \in M'$.

In fact, we may fix $x \in M'$ and assume that $b(x) \neq 0$. For $\beta = (d\chi/d\tau)\,\rho - (d\sigma/d\tau)\,\omega$, (6.4.ii) gives $\beta \wedge d\tau = 0$, and so, locally, $\beta = \xi \wedge d\tau$ for some 1-form ξ. As β and our b satisfy (5.2) (cf. (4.1)), ξ and $d\tau$ are linearly independent at x. By (2.7.i), $\imath_w\beta = \xi(w)\, d\tau - (d_w\tau)\xi$ for any vector w. Thus, the null-space $\operatorname{Ker}\beta(x) = \{w \in T_xM : \imath_w\beta = 0\}$ is the intersection of the kernels of ξ and $d\tau$ at x, that is, $[\operatorname{Ker}\beta(x)]^\perp = \operatorname{Span}\{v(x), u'(x)\}$, where $v = \nabla\tau$ and u' is the vector field with $\xi = g(u', \cdot\,)$. Since g, r, are Hermitian (cf. § 4), so is our b, and, consequently, both $\operatorname{Ker} b(x)$ and $[\operatorname{Ker} b(x)]^\perp$ are J-invariant, while, by (5.2), $\operatorname{Ker}\beta(x) = \operatorname{Ker} b(x)$ (the null-space of $b(x)$). As $v(x) \in [\operatorname{Ker} b(x)]^\perp = \operatorname{Span}\{v(x), u'(x)\}$, we now have $[\operatorname{Ker} b(x)]^\perp = \operatorname{Span}\{v(x), u(x)\}$ with $u = Jv$, as required.

As all non-zero vectors in $\mathcal{H}$ are eigenvectors of $b = (d\chi/d\tau)\,\mathrm{r} - (d\sigma/d\tau) g$ for the eigenvalue 0, at points $x \in M'$ where $d\chi \neq 0$ they are eigenvectors of r for the eigenvalue $\lambda(x)$, with $\lambda = (d\sigma/d\tau)/(d\chi/d\tau) = d\sigma/d\chi$ (so that $d\sigma = \lambda\, d\chi$). Now (6.3) implies the same for $\nabla d\tau$ (with some eigenvalue $\phi(x)$) instead of r and $\lambda(x)$. Consequently, $b = \mathrm{r}$ and $b = \nabla d\tau$ both satisfy (7.2.b), and hence (by Remark 7.1) also (7.2.a), at every such x, which completes the proof. ■

Corollary 9.2. *Let a non-constant C^∞ function $\tau : M \to \mathbf{R}$ on a Kähler manifold (M, g) of complex dimension $m \geq 2$ satisfy* (6.3) *with $\chi, \sigma : M \to \mathbf{R}$ such that $d\sigma \wedge d\tau = d\chi \wedge d\tau = 0$ and $d\chi \neq 0$ wherever $d\tau \neq 0$. Then τ is a special Kähler-Ricci potential on (M, g), that is, satisfies* (7.1) *as well.*

This is clear from Lemmas 6.1 and 9.1 along with Remark 7.3. ■

Corollary 9.3. *Condition* (0.1) *with $m \geq 3$, or* (0.2) *with $m = 2$, implies* (7.1).

In fact, (6.2) and Propositions 6.4 and 6.5 yield the hypotheses of Corollary 9.2. ■

Remark 9.4. In terms of the septuple $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ associated with a special Kähler-Ricci potential τ on a Kähler manifold of complex dimension $m \geq 2$ (Definition 7.2), condition (10.3.iii) below is necessary and sufficient for M, g, m, τ to satisfy (0.1). In fact, by (7.4), the first equality in (6.2), with $n = 2m$, is equivalent to $2(m-1)\phi + \tau\lambda = (n-2)\sigma = 2(m-1)\psi + \tau\mu$.

10. Differential equations related to (7.1) and (0.1)

For a fixed integer $m \geq 2$, let the system of ordinary differential equations

$$
\begin{aligned}
&dQ = 2\psi\, d\tau\,, \qquad dY = -2\mu\, d\tau\,, \qquad Q\, d\phi = 2(\psi-\phi)\phi\, d\tau\,,\\
&Q\, d\psi = [2(m-1)(\phi-\psi)\phi - \mu Q]\, d\tau\,, \qquad Q\, d\lambda = 2(\mu-\lambda)\phi\, d\tau\,,
\end{aligned}
\tag{10.1}
$$

$$
\begin{aligned}
\text{i)}\quad & 2(m-1)(\psi-\phi)Q\, d\mu\\
&\qquad = (\lambda-\mu)\left[\lambda Q + (2m-3)\mu Q + 4(m-1)^2(\psi-\phi)\phi\right] d\tau\,,\\
\text{ii)}\quad & (m-1)(\psi-\phi)\, d\mathrm{s} = (\lambda-\mu)[\lambda + (2m-3)\mu]\, d\tau\,,
\end{aligned}
\tag{10.2}
$$

be imposed on seven unknown C^1 functions $Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu$ of the real variable τ, defined on an unspecified interval. We will also require that, on this interval,

$$
\begin{aligned}
\text{i)}\quad & Q \neq 0 \quad \text{everywhere},\\
\text{ii)}\quad & Y = 2\psi + 2(m-1)\phi\,, \qquad \mathrm{s} = 2\mu + 2(m-1)\lambda\,,\\
\text{iii)}\quad & 2(m-1)(\psi-\phi) = (\lambda-\mu)\tau.
\end{aligned}
\tag{10.3}
$$

The constraints (10.3.ii), (10.3.iii) amount to vanishing of specific *integrals* of the system (10.1)–(10.2); see Lemma 10.1 below.

Conditions (10.1)–(10.3) arise in our discussion of (7.1) and (0.1) (see Lemma 11.1 and Corollary 11.2). Certain particularly simple C^∞ solutions to (10.1)–(10.3) have $\phi = 0$ identically (cf. Corollary 13.2(ii), (iii)). These special solutions will be classified in § 19, while in § 20 and § 21 we deal with the opposite case:

$$
\phi \neq 0 \quad \text{everywhere in the interval.}
\tag{10.4}
$$

Lemma 10.1. *Let $m \geq 1$ be a fixed integer. Then both $2c = 2\tau - Q/\phi$ and $2\psi + 2(m-1)\phi - Y$ are constant, as functions of τ, on any interval on which $Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu$ are C^1 functions satisfying* (10.3.i), (10.4) *and the first four equations in* (10.1). *Similarly, relations* (10.1)–(10.3.i)*, with $m \geq 2$, imply constancy of $2\mu + 2(m-1)\lambda - \mathrm{s}$ on any interval on which $\phi \neq \psi$, and constancy of $Z - \tau$, with $Z = 2(m-1)(\psi-\phi)/(\lambda-\mu)$, on any interval on which $\lambda \neq \mu$.*

In fact, the second assertion follows since (10.1)–(10.3.i) yield, for $m \geq 2$,

$$
Z\, d[\mu + (m-1)\lambda]/d\tau = \lambda + (2m-3)\mu \quad \text{and} \quad dZ/d\tau = 1 \quad \text{wherever} \quad \lambda \neq \mu,
\tag{10.5}
$$

while the first four equations in (10.1) with $m \geq 1$ give $d[Q/\phi]/d\tau = 2$ wherever $\phi \neq 0$ and $d[\psi + (m-1)\phi]/d\tau = -\mu$ wherever $Q \neq 0$, proving the first claim. ■

For further integrals of the system (10.1)–(10.4), see § 27.

Remark 10.2. For a septuple $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ of C^1 functions of a variable τ, (10.1) and (10.3) with $m \geq 1$ imply (10.2). In fact, applying d to (10.3.iii), then multiplying by $(\lambda-\mu)Q$ and, finally, replacing: $(\lambda-\mu)\tau$ with $2(m-1)(\psi-\phi)$ (cf. (10.3.iii)), and $d\lambda$, $d\phi$, $d\psi$ with what is provided by (10.1), we obtain (10.2.i). Now (10.2.ii) is clear from (10.2.i) and the formulae for $d\lambda$ and s in (10.1), (10.3.ii).

11. Equations (10.1) for special Kähler-Ricci potentials

Lemma 11.1. *Let* $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ *be the septuple associated with a special Kähler-Ricci potential* τ *on a Kähler manifold* (M, g) *of complex dimension* $m \geq 1$*, as in* Definition 7.2*. Then, in the open set* $M' \subset M$ *where* $d\tau \neq 0$*,*

(a) *each of the five functions* Q, Y, ϕ, ψ, μ *is, locally, a* C^∞ *function of* τ*,*
(b) $Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu$ *satisfy* (10.1.i)–(10.1.iv)*,* (10.3.i) *and* (10.3.ii)*,*

where (10.1.i)–(10.1.v) *are the five equations forming* (10.1)*. If, in addition,* $m \neq 2$*, then* s, λ *also are, locally in* M'*, functions of* τ*, and* (10.1.v) *holds.*

Proof. On M', Lemma 7.5 gives (10.1.i), (10.1.ii), which implies (a) for Q, Y, and hence also for ψ, μ (as $2\psi = dQ/d\tau$, $2\mu = -dY/d\tau$), and ϕ (cf. Lemma 7.5(i); by Definition 7.2, $\phi = \lambda = 0$ when $m = 1$). This proves (a) for all five functions, and (combined with Lemma 7.5) also the final clause.

Let us now set $v = \nabla\tau$ and $b = \nabla d\tau$, As (2.9.c), (5.4) and (10.1.ii) give $\delta b = -\mu\, d\tau$ for $b = \nabla d\tau$, (2.4.i) yields $\imath_v(\delta b) = -\mu\imath_v d\tau = -\mu Q$, while, by (7.4), $\langle b, b\rangle = 2\psi^2 + 2(m-1)\phi^2$, with $\langle\,,\rangle$ as in (2.5). However, $\delta(\nabla_v v) = \imath_v(\delta b) + \langle b, b\rangle$ due to the obvious local-coordinate identity $[\tau^{,k}\tau_{,jk}]^{,j} = \tau^{,k}\tau_{,jk}{}^{j} + \tau_{,jk}\tau^{,jk}$. Thus, $\delta(\nabla_v v) = -\mu Q + 2\psi^2 + 2(m-1)\phi^2$ while, as $Y = \Delta\tau$ and $\nabla_v v = \psi v$ (Lemma 7.5(ii)), $\delta(\nabla_v v) = \delta(\psi v) = d_v\psi + \psi Y = d_v\psi + [2\psi + 2(m-1)\phi]\psi$ (by (2.8.ii), (2.8.i) and Lemma 7.5(i)). Equating the two formulae for $\delta(\nabla_v v)$, we obtain $d_v\psi = 2(m-1)(\phi - \psi)\phi - \mu Q$. Both sides of (10.1.iv) thus yield the same value under $\imath_v$ (cf. (2.4.i)) and, since they both are functional multiples of $d\tau$ (by (a)), while $\imath_v d\tau = Q \neq 0$ by (2.4.i), they must coincide. This proves (10.1.iv). Since $(1-m)\,d\phi = d\psi + \mu\, d\tau$ (as one sees evaluating dY from Lemma 7.5(i) and using (10.1.ii)), (10.1.iii) now is immediate from (10.1.iv). Finally, (10.3.i) follows since $Q = g(\nabla\tau, \nabla\tau) > 0$ on M', while (10.3.ii) is obvious from Lemma 7.5(i), which completes the proof. ∎

Corollary 11.2. *Let* M, g, m, τ *satisfy* (0.1) *with* $m \geq 3$*, or* (0.2) *with* $m = 2$*, and let* $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ *be the septuple associated with* τ*; cf.* Definition 7.2 *and* Corollary 9.3*. Then* (10.1)–(10.3) *hold in the open set on which* $d\tau \neq 0$*.*

In fact, τ satisfies (7.1) (Corollary 9.3). Now Lemma 11.1 and Remark 9.4 give (10.1) and (10.3), except for the last formula in (10.1) when $m = 2$. That formula then follows, however, from the expression for $(m-2)\nabla\lambda$ in Lemma 7.5, in which we now have $[\nabla\lambda]^{\mathrm{vrt}} = \nabla\lambda$, as λ is (locally, at points with $d\tau \neq 0$) a function of τ, since this is true of s (Proposition 6.5) and μ (Lemma 11.1), while $2(m-1)\lambda = \mathrm{s} - 2\mu$ by Lemma 7.5(i). Now Remark 10.2 yields (10.2). ∎

12. The ϕ alternative

The following two lemmas are well-known.

Lemma 12.1. *Let* u *be a Killing field on a Riemannian manifold* (M, g)*, and let* $y \in M$ *be a point such that* $u(y) = 0$*.*

(i) *For every sufficiently small* $\mathrm{d} > 0$*, the flow of* u *restricted to the radius* d *open ball* U *centered at* y *consists of 'global' isometries* $U \to U$*.*

(ii) *If* d *and* U *in* (i) *are chosen so that, in addition, the exponential mapping* $\exp_y : U' \to U$ *is a diffeomorphism, where* U' *is the radius* d *open ball in* T_yM *centered at* 0*, then* u *restricted to* U *is the* $\exp_y$*-image of the linear vector field on* U' *given by* $w \mapsto \nabla_w u$*.*

This is clear as the diffeomorphism $\exp_y : U' \to U$, for any small d, makes every isometry $\Phi : U \to U$ with $\Phi(y) = y$ correspond to the linear mapping $d\Phi_y : U' \to U'$. Now (ii) is immediate if we apply this to the isometries forming the flow of u and note that their differentials at y form a one-parameter group of linear isometries with the infinitesimal generator $(\nabla u)(y)$. ■

Lemma 12.2. *For a Killing vector field* u *on a Riemannian manifold* (M, g)*, let* $N(u) = \{y \in M : u(y) = 0\}$ *be the set of all zeros of* u*. If* $u \neq 0$ *somewhere in* M*, then, for every connected component* N *of* $N(u)$*,*

(a) N *is contained in an open set that does not intersect any other component,*
(b) $N \subset M$ *is a closed set and a submanifold with the subset topology,*
(c) *the submanifold* N *is totally geodesic in* (M, g) *and* $\dim M - \dim N \geq 2$*,*
(d) *for any* $y \in N$ *we have* $T_yN = \mathrm{Ker}\,[(\nabla u)(y)] = \{w \in T_yM : \nabla_w u = 0\}$*.*

Furthermore, the set $M' = M \smallsetminus N(u)$ *is connected, open and dense in* M*.*

In fact, (a)–(d) (except for the inequality in (c)) are immediate from Lemma 12.1(ii). Now let us fix $y \in M$ with $u(y) = 0$ and d, U', U with (i), (ii) of Lemma 12.1. The space $V = \mathrm{Ker}\,[(\nabla u)(y)] \subset T_yM$ has $\dim V < \dim M$, or else u and ∇u would both be zero at y, that is, u would vanish identically (Remark 5.1). Next, choosing $w \in V^\perp \subset T_yM$ with $w \neq 0$, we have $\nabla_w u \in V^\perp \cap w^\perp$ (as $(\nabla u)(y)$ is skew-adjoint) and $\nabla_w u \neq 0$ (since $w \notin V$). Hence $w, \nabla_w u \in V^\perp$ are non-zero and orthogonal to each other, which yields $\dim M - \dim V \geq 2$. As $U \cap N(u) = \exp_y(U' \cap V)$ (by Lemma 12.1(ii)), we thus obtain $\dim M - \dim N \geq 2$, which (along with (a)) gives connectedness and denseness of $M' = M \smallsetminus N(u)$. ■

Corollary 12.3. *Let* τ *be a non-constant Killing potential on a Kähler manifold; cf.* § 5*. Then the open subset* M' *on which* $d\tau \neq 0$ *is connected and dense in* M*.*

This is obvious from Lemma 12.2, since $u = J(\nabla\tau)$ is a Killing field. ■

Lemma 12.4. *Let* τ *satisfy* (7.1) *on a Kähler manifold* (M, g) *of complex dimension* $m \geq 1$*, and let* $(0, L) \ni s \mapsto x(s) \in M'$ *be an integral curve of the vector field* $\pm v/|v|$*, where* $\pm$ *is some sign,* $v = \nabla\tau$ *and* $M' \subset M$ *is the open set on which* $d\tau \neq 0$*. Then, for the eigenvalue function* ϕ *appearing in* (7.4)*, either* $\phi = 0$ *at* $x(s)$ *for all* $s \in (0, L)$*, or* $\phi \neq 0$ *at all* $x(s)$ *with* $s \in (0, L)$*.*

In fact, let I' be a maximal (non-empty) subinterval of $(0, L)$ with $\phi \neq 0$ at all $x(s)$ with $s \in I'$. By Lemmas 11.1(b) and 10.1, there exists a constant c such that $Q = 2(\tau - c)\phi$ at every $x(s)$ with $s \in I'$, where $Q = g(v, v)$. Hence, if such a subinterval exists, it must coincide with $(0, L)$, or else it would have an endpoint $L' \in (0, L)$, with $Q = \phi = 0$ at $x(L')$, contrary to our assumption that $x(s) \in M'$, that is, $v \neq 0$ at $x(s)$, for all $s \in (0, L)$. ■

Lemma 12.5. *Given τ with* (7.1) *on a Kähler manifold (M,g) of complex dimension $m \geq 1$, let $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ be as in* Definition 7.2*, and let $M' \subset M$ be the open set on which $d\tau \neq 0$. Then either $\phi = 0$ identically on M', or $\phi \neq 0$ everywhere in M'. In the latter case, there exists a constant c such that relations $Q/\phi = 2(\tau - c)$ and $\tau \neq c$ hold everywhere in M'.*

In fact, given $x \in M'$ with $\phi(x) = 0$, let P be the connected component, containing x, of the preimage of $\tau(x)$ under $\tau : M' \to \mathbf{R}$. Thus, $\phi = 0$ on P since $\phi(x) = 0$ and, by Lemma 11.1(a), ϕ restricted to P is constant. Lemma 12.4(b) now implies that $\phi = 0$ along every integral curve of $v = \nabla\tau$ in M' which intersects P. As P is a submanifold of M', while $v = \nabla\tau$ is normal to P and nonzero everywhere in M', the union of these integral curves contains a neighborhood of x in M'. Thus, the set of all $x \in M'$ with $\phi(x) = 0$ is not only (relatively) closed, but also open in M'. Our either-or claim now follows from connectedness of M' (Corollary 12.3). Finally, if $\phi \neq 0$ everywhere in M', Lemmas 11.1(b) and 10.1 show that the required constant c exists, locally, in M'. As M' is connected (Corollary 12.3), c has a unique value throughout M'. ■

13. Special Kähler-Ricci potentials and reducibility

Lemma 13.1. *Suppose that $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ is the septuple associated, as in* Definition 7.2*, with a special Kähler-Ricci potential τ on a Kähler manifold (M,g) of complex dimension $m \geq 2$, and $v, u, \mathcal{V}, \mathcal{H}$ are given by* (7.3)*, while w, w' are C^1 vector fields, defined in the open set M' on which $d\tau \neq 0$ and orthogonal to v and u. Denoting by $[\nabla_w w']^{\mathrm{vrt}}$ and $[w, w']^{\mathrm{vrt}}$ the $\mathcal{V}$ components of $\nabla_w w'$ and the Lie bracket $[w, w']$ relative to the decomposition $TM' = \mathcal{H} \oplus \mathcal{V}$, we have*

$$\text{(13.1)}\quad \begin{aligned} &\text{a) } Q[\nabla_w w']^{\mathrm{vrt}} = -\phi[g(w,w')v + \omega(w,w')u] \quad \text{and} \\ &\text{b) } Q[w,w']^{\mathrm{vrt}} = -2\phi\,\omega(w,w')u, \ \text{ where } \omega \text{ is the Kähler form of } (M,g), \\ &\text{c) } g(\nabla_w v, w') = \phi g(w,w'), \quad g(\nabla_w u, w') = \phi\omega(w,w'). \end{aligned}$$

If, in addition, w, w' commute with v and u, then

$$\text{(13.2)}\quad \begin{aligned} &\text{i) } \nabla_v w = \nabla_w v = \phi w, \quad \nabla_u w = \nabla_w u = \phi J w, \\ &\text{ii) } d_v \varPsi = d_u \varPsi = 0 \qquad \text{for} \quad \varPsi = \phi g(w,w')/Q. \end{aligned}$$

Proof. By (2.3) and (7.4), $g(\nabla_w v, w') = (\nabla d\tau)(w, w') = \phi g(w, w')$, while (2.3), (7.4) and (4.1) yield $g(\nabla_w u, w') = g(\nabla_w (Jv), w') = g(J(\nabla_w v), w') = -g(\nabla_w v, Jw')$ $= -(\nabla d\tau)(w, Jw') = -\phi g(w, Jw') = \phi\omega(w, w')$, which proves (13.1.c).

Since (13.2.ii) is obvious when $\phi = 0$ identically, let us assume (cf. Lemma 12.5) that $\phi \neq 0$ everywhere in M'. By (2.1.ii), $g(\nabla_v w, w') = g(\nabla_w v, w')$ and $g(\nabla_u w, w') = g(\nabla_w u, w')$ whenever w, w' commute with v and u, so that (13.1.c) implies $d_v \varUpsilon = 2\phi\varUpsilon$ and $d_u \varUpsilon = 0$ for the function $\varUpsilon = g(w, w')$. Also, by Lemma 12.5, $d(\phi/Q) = -(\phi/Q)^2 d(Q/\phi) = -2(\phi/Q)^2 d\tau$, so that $d_v(\phi/Q) = -2\phi^2/Q$ and $d_u(\phi/Q) = 0$ since $d_v\tau = Q$ and $d_u\tau = g(u, v) = 0$ (see (2.4.i), (7.5)). As $\varPsi = (\phi/Q)\varUpsilon$ in (13.2.ii), our formulae for $d_v\varUpsilon, d_u\varUpsilon$ give (13.2.ii).

Next, $g(v, \nabla_w w') = -g(\nabla_w v, w')$ since $g(v, w') = 0$ (and similarly for u instead of v). This proves (13.1.a); in fact, (13.1.c) and (7.5) now show that both sides of (13.1.a) have the same inner product with u and v, while $\mathcal{V} = \mathrm{Span}\,\{v, u\}$. Thus, (13.1.b) is immediate from (13.1.a) and (2.1.ii).

Finally, for w, w' orthogonal to v, u and commuting with them, $\nabla_v w$ and $\nabla_u w$ are orthogonal to v and u (by the Leibniz rule, cf. the last paragraph, and Lemma 7.5(ii)), while $\nabla_v w = \nabla_w v$, $\nabla_u w = \nabla_w u$ by (2.1.ii). Hence (13.2.i) follows, as one sees taking the inner products of the quantities involved with any vector field w' orthogonal to v and u and using (13.1.c). This completes the proof. ■

Corollary 13.2. *Let τ satisfy* (7.1) *on a Kähler manifold (M, g) of complex dimension $m \geq 2$, and let $M' \subset M$ be the open set of points at which $d\tau \neq 0$. For the distributions $\mathcal{V}, \mathcal{H}$ and the function ϕ on M' given by* (7.3)–(7.4)*,*

- (i) *$\phi = 0$ identically on M' if and only if $\mathcal{H}$ is integrable,*
- (ii) *if $\phi = 0$ identically on M', then, locally, (M', g) is a Riemannian product with $\mathcal{H}, \mathcal{V}$ serving as the factor distributions,*
- (iii) *conversely, if there exists a non-empty open connected set $U \subset M'$ such that (U, g) is a non-Einstein, Riemannian-and-Kähler product of two lower-dimensional Kähler manifolds, then $\phi = 0$ identically on M'.*

In fact, (13.1.b) implies (i). If $\phi = 0$ on M', then $\mathcal{V}$ is parallel since, by Lemma 7.5(ii) and (13.2.i), the covariant derivatives of $v = \nabla\tau$ and $u = Jv$ in all directions lie in $\mathcal{V}$. This yields (ii). Finally, making U in (iii) smaller, we may assume (7.4) with $\lambda \neq \mu$ everywhere in U (so that r is not a multiple of g at any point of U). Then $\mathcal{V}$ restricted to U is an eigenspace distribution of complex dimension 1 for r, and so $\mathcal{V}$ must be contained in one factor distribution of the product decomposition in (iii). Now (13.1.a) with $w = w' \neq 0$ tangent to the other factor gives $\phi = 0$ on U, and hence on M' (see Lemma 12.5). ■

14. Submersions and horizontal vectors

Given manifolds M, N and a C^∞ submersion $\pi : M \to N$, we will refer to the subbundle $\mathcal{V} = \mathrm{Ker}\, d\pi$ of TM as the *vertical distribution* of π. A fixed Riemannian metric g on M then gives rise to the *horizontal distribution* $\mathcal{H} = \mathcal{V}^\perp$, and we denote by $w^{\mathrm{hrz}}, w^{\mathrm{vrt}}$ the $\mathcal{H}$ and $\mathcal{V}$ components of any vector (field) w tangent to M relative to the decomposition $TM = \mathcal{H} \oplus \mathcal{V}$. Also, the Levi-Civita connection ∇ of g induces connections $\nabla^{\mathrm{hrz}}, \nabla^{\mathrm{vrt}}$ in the vector bundles $\mathcal{H}, \mathcal{V}$ over M, with

$$(14.1) \qquad \nabla^{\mathrm{hrz}}_w w' = [\nabla_w w']^{\mathrm{hrz}}, \qquad \nabla^{\mathrm{vrt}}_w w' = [\nabla_w w']^{\mathrm{vrt}}$$

for vector fields w and horizontal/vertical C^1 vector fields w' on M.

Remark 14.1. Given a Riemannian manifold (M, g) and a *surjective* submersion $\pi : M \to N$, we will use the horizontal-lift operation to identify vector fields on N with those horizontal vector fields on M (that is, sections of $\mathcal{H}$) which are π-projectable. This does not usually lead to notational confusion, with one notable exception. Namely, the Lie bracket of two π-projectable vector fields is π-projectable onto the Lie bracket of their π-images (see [**18**, p. 10]), but Lie brackets of horizontal fields need not be horizontal. Also, all vertical vector fields are π-projectable (onto 0). Thus, if $[w,w']$ is the Lie bracket in M of any π-projectable horizontal C^∞ vector fields w,w' on M, then, under our identification,

$$[w,w']^{\mathrm{hrz}} \text{ corresponds to the Lie bracket of } w \text{ and } w' \text{ in } N. \tag{14.2}$$

We will encounter cases in which (M,g) is a Riemannian manifold with a submersion $\pi : M \to N$ and, in addition, there are given objects u, β, Θ such that

(14.3)
- a) u is a vertical vector field on M and β is a differential 2-form on M,
- b) $\Theta : \mathcal{H} \to \mathcal{H}$ is a vector-bundle morphism, with $\mathcal{H}$ as above, while
- c) $[w,w']^{\mathrm{vrt}} = \beta(w,w')u$ for all horizontal C^1 vector fields w,w', and
- d) $[\nabla_u w]^{\mathrm{hrz}} = \Theta w$ for all π-projectable horizontal C^1 vector fields w.

Conditions (14.3.c)–(14.3.d) are geometrically natural. In (14.3.c), the dependence of $[w,w']^{\mathrm{vrt}}$ on w,w' is pointwise, that is, the value of $[w,w']^{\mathrm{vrt}}$ at any $x \in M$ depends only on $w(x)$ and $w'(x)$, as one sees multiplying w or w' by a function (cf. [**18**, Proposition 3.1 on p. 26]). In (14.3.d), π-projectable horizontal vector fields form a trivializing space of sections in the vector bundle $\mathcal{H}$ restricted to any of the submanifolds $\pi^{-1}(y)$ with $y \in N$, and this is why, for ∇^{hrz} as in (14.1), ∇_u^{hrz} makes sense as a *bundle morphism.* (For details, see Remark 14.2 below.)

Remark 14.2. Let V be a *trivializing space of C^∞ sections* of a vector bundle $\mathcal{E}$ over any base manifold, that is, a vector space of sections such that, for every point y of the base and every $w_0 \in \mathcal{E}_y$, there exists a *unique* $w \in V$ with $w(y) = w_0$. For any fixed connection ∇ in $\mathcal{E}$ and any vector field u on the base, we then may treat ∇_u as a vector-bundle morphism $\mathcal{E} \to \mathcal{E}$, sending any $w_0 \in \mathcal{E}_y$ to $(\nabla_u w)(y)$, with w chosen as above for y and w_0.

15. Horizontally homothetic submersions

A C^∞ submersion $\pi : M \to N$ between Riemannian manifolds (M,g), (N,h) is called *horizontally homothetic* [**15**] if, for some function $f : M \to (0,\infty)$,

$$\pi^* h = g/f \quad \text{on} \quad \mathcal{H} \quad \text{and} \quad [\nabla f]^{\mathrm{hrz}} = 0\,, \tag{15.1}$$

with $...^{\mathrm{hrz}}, \mathcal{H}$ as in § 14, that is, if the pullback tensor $\pi^* h$ agrees with g/f on the horizontal distribution, and f has a vertical gradient. For more on such submersions, including their relation to harmonic morphisms, see [**15**, p. 464].

A submersion π with (15.1) obviously becomes a *Riemannian submersion* if one replaces g with the conformally related metric $\tilde g = g/f$. Conversely, any Riemannian submersion $\pi : M \to N$ between $(M,\tilde g)$ and (N,h) is a horizontally homothetic submersion between (M,g) and (N,h), where $g = f\tilde g$ for any fixed C^∞ function $f : M \to (0,\infty)$ with $[\nabla f]^{\mathrm{hrz}} = 0$. Further examples of (15.1) are therefore provided by projections $\pi : M \to S$ of a product manifold $M = N \times S$ endowed with a *warped product* metric g, obtained from a metric h on N along with some metric on S and a function $S \to (0,\infty)$. See, for example, [**4**] or [**3**, p. 237].

As a result, the next two lemmas could also be obtained by combining known facts about Riemannian submersions and conformal changes of metrics. In particular, (15.2) could in principle be derived from the horizontal curvature formula in [**15**, p. 463]. Such a derivation is, however, more involved than one might expect, since the restriction of the Ricci tensor to the horizontal distribution contains, in addition to horizontal curvature terms, also 'vertizontal' ones. For this reason we prove (15.2) differently. Our argument highlights an appealing feature of Kähler geometry: direct access to the Ricci tensor, bypassing the curvature.

Lemma 15.1. *Let $\pi : M \to N$ be a horizontally homothetic, surjective submersion between Riemannian manifolds (M,g), (N,h), and let ∇ and D stand for the Levi-Civita connections of g and h, respectively. For any π-projectable horizontal C^∞ vector fields w, w' on M treated as vector fields on N (cf.* Remark 14.1*),*

(i) *the vector field $[\nabla_w w']^{\mathrm{hrz}}$ on M is π-projectable,*
(ii) *$\mathrm{D}_w w'$ equals $[\nabla_w w']^{\mathrm{hrz}}$ treated as a vector field on N.*

Proof. Our g, ∇ and any π-projectable horizontal C^∞ vector fields u, v, w in M satisfy (2.1.i) and, since $\mathcal{V}$ and $\mathcal{H}$ are g-orthogonal, (2.1.i) will remain valid when $\nabla_w v$ and all Lie brackets are replaced by their horizontal components. Since f in (15.1) is constant along u, v and w, (2.1.i) will still hold if we further replace every occurrence of g by g/f. Finally, using (14.2) and (15.1) we see that this last modified version of (2.1.i) is also valid with $\mathrm{D}_w v$ instead of $[\nabla_w v]^{\mathrm{hrz}}$ (since (2.1.i) holds for (N,h) as well). Both $\mathrm{D}_w v$, $[\nabla_w v]^{\mathrm{hrz}}$ thus are horizontal vector fields on M having the same inner product with every π-projectable horizontal field, and hence must coincide. This in turn implies (i), and completes the proof. ■

Lemma 15.2. *Let $\mathcal{V}$ and $\mathcal{H}$ be the vertical and horizontal distributions of a holomorphic, surjective, horizontally homothetic submersion $\pi : M \to N$ between Kähler manifolds (M,g) and (N,h), and let* (14.3) *hold for some u, β and a complex-linear morphism $\Theta : \mathcal{H} \to \mathcal{H}$. Denoting by ρ, $\rho^{(h)}$ and Ω the Ricci forms of g, h and, respectively, the curvature form (cf.* Remark 3.1*), of the connection in the highest complex exterior power of $\mathcal{V}$, induced by ∇^{vrt} with* (14.1)*, and letting $\mathrm{Trace}_{\mathbf{C}}\Theta : M \to \mathbf{C}$ be the pointwise trace of Θ, we have*

$$\rho \;=\; \Omega \;+\; \pi^*\rho^{(h)} \;-\; i[\mathrm{Trace}_{\mathbf{C}}\Theta]\beta \qquad \text{on} \quad \mathcal{H}. \tag{15.2}$$

Proof. Let m and q be the complex dimensions of M and N, and let the ranges of indices be: $a, b \in \{1,\ldots,m\}$, $j, k \in \{1,\ldots,q\}$, $\lambda, \mu \in \{q{+}1,\ldots,m\}$. Locally, we can choose π-projectable horizontal vector fields w_j and vertical vector fields w_λ, which together trivialize the complex vector bundle TM on an open set $U \subset M$. Choosing the corresponding 1-forms Γ_a^b as in (3.1.i), we now have $[\nabla_v w_j]^{\mathrm{hrz}} = \Gamma_j^k(v) w_k$ and $[\nabla_v w_\lambda]^{\mathrm{vrt}} = \Gamma_\lambda^\mu(v) w_\mu$ for any vector field v on U (due to (3.1.i) with $[w_j]^{\mathrm{vrt}} = [w_\lambda]^{\mathrm{hrz}} = 0$). Now (14.1) gives $\nabla_v^{\mathrm{vrt}} w_\lambda = \Gamma_\lambda^\mu(v) w_\mu$, and so, for $\hat{w} = w_{q+1} \wedge \ldots \wedge w_m$ we have $\nabla_v^{\mathrm{vrt}} \hat{w} = \Gamma(v)\hat{w}$ with $\Gamma = \Gamma_\lambda^\lambda$. Thus, $\Omega = i\,d\Gamma_\lambda^\lambda$, since $\Omega = i\,d\Gamma$ (Remark 3.1). Note that we sum over repeated indices.

As $[\nabla_v w_j]^{\mathrm{hrz}} = \varGamma_j^k(v)w_k$, Lemma 15.1(i) shows that $\varGamma_j^k(v)$ is π-projectable, that is, descends to a function in N, whenever v is a π-projectable horizontal vector field in M, so that the 1-forms $\varGamma_j^k$ in M, restricted to $\mathcal{H}$, may be viewed as 1-forms in N (cf. Remark 14.1). Lemma 15.1(ii) then gives $\mathrm{D}_v w_j = \varGamma_j^k(v)w_k$, with the π-projectable horizontal vector fields v and w_k in M treated, simultaneously, as vector fields in N. Now, by Lemma 4.2 (for (N,h)), $\rho^{(h)} = i\,d_N\varGamma_j^j$, where d_N is the exterior derivative in N. Note that d_N differs from d, the exterior derivative in M, due to the Lie-bracket term in (2.7.ii), as the M and N Lie brackets in (14.2) differ by $[w,w']^{\mathrm{vrt}}$, that is, by $\beta(w,w')u$, cf. (14.3.c). Thus, $d\varGamma_j^j = d_N\varGamma_j^j - \varGamma_j^j(u)\beta$ on $\mathcal{H}$, while $\varGamma_j^j(u) = \mathrm{Trace}_{\mathbf{C}}\varTheta$, since $\varTheta w_j = [\nabla_u w_j]^{\mathrm{hrz}} = \varGamma_j^k(u)w_k$. Consequently, $d\varGamma_j^j = d_N\varGamma_j^j - [\mathrm{Trace}_{\mathbf{C}}\varTheta]\beta$. However, Lemma 4.2 for (M,g) gives $\rho = i\,d\varGamma_a^a = i\,d\varGamma_j^j + i\,d\varGamma_\lambda^\lambda$. Hence the last relation, along with $\varOmega = i\,d\varGamma_\lambda^\lambda$ and $\rho^{(h)} = i\,d_N\varGamma_j^j$ (see above), yields (15.2). This completes the proof. ■

Corollary 15.3. *Suppose that a surjective holomorphic submersion $\pi : M' \to N$ between Kähler manifolds $(M',g), (N,h)$ of complex dimensions $m \ge 2$ and $m-1$ satisfies* (15.1) *for some $f : M' \to (0,\infty)$. Also, let v be a C^∞ vertical vector field without zeros on M' such that, for all π-projectable horizontal C^∞ vector fields w, w' on M',*

$$\nabla_v v = \psi v, \quad \nabla_u v = \psi u, \quad \nabla_w v = \phi w, \quad \nabla_u w = \phi Jw, \quad \text{with } u = Jv, \tag{15.3}$$

for some functions $\phi, \psi : M' \to \mathbf{R}$. Finally, let (14.3.c) *hold for the 2-form $\beta = -2\varepsilon\omega/f$, where ε is a real constant, $u = Jv$, and ω is the Kähler form of (M',g), with $\ldots^{\mathrm{vrt}}, \ldots^{\mathrm{hrz}}, \mathcal{V}, \mathcal{H}$ as in* § 14.

Denoting by r *and* $\mathrm{r}^{(h)}$ *the Ricci tensors of g and h, we have the following*:

(i) *If* $\mathrm{r}^{(h)} = \kappa h$ *for a function $\kappa : N \to \mathbf{R}$, then* $\mathrm{r} = \lambda g$ *on $\mathcal{H}$, with $\lambda = (\kappa - \varepsilon Y)/f$ and $Y = 2\psi + 2(m-1)\phi$.*

(ii) *If* $\mathrm{r} = \lambda g$ *on $\mathcal{H}$ for a function $\lambda : M' \to \mathbf{R}$, then* $\mathrm{r}^{(h)} = \kappa h$ *with $\kappa : N \to \mathbf{R}$ given by $\kappa = \lambda f + \varepsilon Y$, where $Y = 2\psi + 2(m-1)\phi$.*

Proof. Our v is a global trivializing section of the complex line bundle $\mathcal{V}$ over M', while (14.1) and (15.3) give $\nabla_w^{\mathrm{vrt}} v = \varGamma(w)v$ for all vectors w, where $\varGamma$ is the complex-valued 1-form with $\varGamma(v) = \psi$, $\varGamma(u) = i\psi$, and $\varGamma(w) = 0$ if w is horizontal. (Note that the dependence of $\nabla_w^{\mathrm{vrt}} v$ on w is pointwise.) Using (2.7.ii) for $\xi = \varGamma$ and (14.3.c) we now get $(d\varGamma)(w,w') = 2\varepsilon i\psi\,\omega(w,w')/f$ for all horizontal vectors w,w'. Also, verticality of v and J-invariance of $\mathcal{H}$, along with our assumptions (14.3.c) and (15.3) show that conditions (14.3) are satisfied by our β with $u = Jv$ and $\varTheta = \phi J$ (so that $\varTheta$ multiplies sections of $\mathcal{H}$ by the function $i\phi$). Lemma 15.2 now gives (15.2), which here reads $\rho - \pi^*\rho^{(h)} = -\varepsilon Y\omega/f$ on $\mathcal{H}$, for $Y = 2\psi + 2(m-1)\phi$. In fact, $\varOmega = i\,d\varGamma$ (see Remark 3.1; the 'highest complex exterior power' of $\mathcal{V}$ required in Lemma 15.2 now is $\mathcal{V}$ itself), so that $\varOmega = -2\varepsilon\psi\omega/f$ on $\mathcal{H}$ (by the above formula for $(d\varGamma)(w,w')$), while $\mathrm{Trace}_{\mathbf{C}}\varTheta = (m-1)i\phi$, that is, $-i[\mathrm{Trace}_{\mathbf{C}}\varTheta]\beta = -2(m-1)\varepsilon\phi\omega/f$. Hence $\rho - \pi^*\rho^{(h)} = -\varepsilon Y\pi^*\omega^{(h)}$ on $\mathcal{H}$, where $\omega^{(h)}$ is the Kähler form of (N,h). (This is clear since $\pi^*\omega^{(h)} = \omega/f$ on $\mathcal{H}$, due to (4.1) and (15.1).)

Assertions (i) and (ii) now follow from the two formulae for $\rho - \pi^*\rho^{(h)}$, as $\pi^*\omega^{(h)} = \omega/f$ on $\mathcal{H}$, while conditions $\mathrm{r} = \lambda g$ (on $\mathcal{H}$) and $\mathrm{r}^{(h)} = \kappa h$ are equivalent to $\rho = \lambda\omega$ (on $\mathcal{H}$) and, respectively, $\rho^{(h)} = \kappa\omega^{(h)}$. This completes the proof. ■

16. Proof of the claim made in § 8

Let the data (8.1) satisfy the conditions listed in § 8. Then g defined in § 8 is a Kähler metric on the complex manifold M', and τ, as a function $M' \to \mathbf{R}$, is a special Kähler-Ricci potential on (M', g) (see (7.1)). Also, for the septuple $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ associated with τ as in Definition 7.2, and the Ricci tensor $\mathrm{r}^{(h)}$ of h,

(a) $Q = a^2r^2\theta$, and $d\tau \neq 0$ everywhere in M',
(b) $\lambda = (\kappa - \varepsilon Y)/f$, that is, $\kappa = \lambda f + \varepsilon Y$, with $\kappa : N \to \mathbf{R}$ such that $\mathrm{r}^{(h)} = \kappa h$ (we identify κ with the composite $\kappa \circ \pi$, that is, view it as a function on M'),
(c) either $\phi = 0$ identically and $\varepsilon = 0$, or $\phi \neq 0$ everywhere, $\varepsilon = \pm 1$ and c in (7.1) is the same as in Lemma 12.5, so that $Q/\phi = 2(\tau - c)$.

This can be verified as follows. We denote by v, u the vector fields (3.3) on $\mathcal{L}$, with our fixed $a \neq 0$. Also, throughout this section, w, w' stand for any two C^∞ vector fields in N and, simultaneously, for their horizontal lifts to $\mathcal{L}$, while the symbol J is used for the complex structure tensors of both N and $\mathcal{L}$. As $\mathcal{H}$ is J-invariant and $\pi : \mathcal{L} \to N$ is holomorphic, Jw means the same, whether treated as a vector field in N, or as a π-projectable horizontal field in $\mathcal{L}$.

It follows now that $v = \nabla\tau$, that is, v is the g-gradient of τ, and

$$
\begin{aligned}
&\nabla_v v = -\nabla_u u = \psi v\,, \quad \nabla_v u = \nabla_u v = \psi u\,, \quad \nabla_v w = \nabla_w v = \phi w\,,\\
&\nabla_u w = \nabla_w u = \phi Jw\,, \quad \nabla_w w' = \mathrm{D}_w w' - \varepsilon[h(w,w')v + h(Jw,w')u]\,,
\end{aligned}
\tag{16.1}
$$

where ∇, D are the Levi-Civita connections of g and h, while $\psi, \phi : M' \to \mathbf{R}$ are given by $2\theta\psi/a = 2\theta + r\,d\theta/dr$, $2f\phi/a = r\,df/dr$. In fact, as $d\tau/dr = ar\theta$ in § 8, (3.4) gives $d_v\tau = Q$ with $Q = a^2r^2\theta$ and $d_u\tau = d_w\tau = 0$. Formulae $g(v,v) = g(u,u) = Q$ and $g(w,w') = fh(w,w')$, $g(v,u) = g(v,w) = g(u,w) = 0$ (due to $\langle v,v\rangle = \langle u,u\rangle = a^2r^2$ and $\mathrm{Re}\langle v,u\rangle = 0$ in Remark 3.2) now give $v = \nabla\tau$ (by showing that $v - \nabla\tau$ is g-orthogonal to v, u and all w), and hence assertion (a) follows. As J is skew-adjoint and $f\phi = \varepsilon Q$ (since $df/d\tau = 2\varepsilon$ and $d\tau/dr = ar\theta$), they also imply that the connection ∇ *defined* by (16.1) makes g parallel. Next,

$$
[w, w']^{\mathrm{vrt}} = -2\varepsilon\omega^{(h)}(w,w')u = -2\varepsilon h(Jw,w')u\,,
\tag{16.2}
$$

$\omega^{(h)}$ being the Kähler form of h. (This is clear from (3.6), (3.3), the assumption about the curvature form in § 8, and (4.1).) Also, $[v,u] = [v,w] = [u,w] = 0$, since v, u and all w are obviously invariant under fibrewise multiplications by complex constants, which constitute the flows of v and u in $\mathcal{L}$. Therefore, by (2.1.ii), (14.2) and (16.2), ∇ defined by (16.1) is torsion-free and so, as $\nabla g = 0$, it must coincide with the Levi-Civita connection of g.

Since $u = Jv$, by (3.3), and h is Hermitian (cf. § 4), (16.1) now shows that the complex structure tensor J in M' is ∇-parallel, that is, commutes with ∇_v, ∇_u and all ∇_w. Thus, g is a Kähler metric on M'.

Remark 2.1 and (16.1) now imply the part of (7.4) involving $\nabla d\tau$, with our $\mathcal{V}, \mathcal{H}$ and ψ, ϕ as in (16.1). The eigenspaces of $\nabla d\tau$ at every point thus are J-invariant and g-orthogonal to one another, so that $\nabla d\tau$ is Hermitian, that is, τ is a non-constant Killing potential on (M', g) (Lemma 5.2(iii)).

Since our $v, u, \mathcal{V}, \mathcal{H}$ obviously satisfy (7.3), all that remains to be verified is the part of (7.4) involving r (cf. Remark 7.3), that is, (7.2.a) for $b = \mathrm{r}$, at every point $x \in M'$, as well as relations (b), (c) at the beginning of this section.

First, setting $Y = \Delta\tau$ we have $Y = \delta v = \mathrm{Trace}\,\nabla v$ (see (2.8.i)), and so, by (16.1), $Y = 2\psi + 2(m-1)\phi$. Thus, Y is a function of r, that is, of τ (as $d\tau/dr = ar\theta \neq 0$ in § 8) and, by (5.4), $\imath_v \mathrm{r} = \mu\, d\tau = \mu\, \imath_v g$, with μ given by $2\mu = -dY/d\tau$. Consequently, $v(x)$ is, at every point $x \in M'$, an eigenvector of $\mathrm{r}(x)$ for the eigenvalue $\mu(x)$. Hermitian symmetry of r (see § 4) now implies that $u(x) = Jv(x)$ is also an eigenvector of $\mathrm{r}(x)$ for the same eigenvalue $\mu(x)$.

The hypotheses of Corollary 15.3 are now satisfied by f, ϕ, ψ defined above. In fact, (15.3) (or, (15.1)) is a part of (16.1) (or, of the definition of g), ∇f being vertical since so is $\nabla\tau = v$ and f is a (linear) function of τ. Also, (14.3.c) with $\beta = -2\varepsilon\omega/f$ is clear from (16.2), since relation $g = f\pi^*h$ on $\mathcal{H}$ and (4.1) give $\pi^*\omega^{(h)} = \omega/f$ on $\mathcal{H}$. Moreover, the premise of Corollary 15.3(i) is satisfied (cf. § 8), and hence so is its conclusion, that is, $\mathrm{r} = \lambda g$ on $\mathcal{H}$ for $\lambda = (\kappa - \varepsilon Y)/f$. Combined with the last paragraph, this yields (7.2.a) for $b = \mathrm{r}$, as well as (b). Finally, by Remark 2.1, ϕ in (7.4) coincides with ϕ in (16.1), so that (as $df/d\tau = 2\varepsilon$ and $d\tau/dr = ar\theta$) ϕ and ε are related as stated in (c), and the remainder of (c) follows from (a) with $f\phi = \varepsilon Q$ and $f = 2\varepsilon(\tau - c)$. ■

Remark 16.1. The construction in § 8 may be conveniently rewritten using a modified version of the data (8.1), in which the initial quintuple $\mathcal{J}, r, \theta, \tau, f$ is replaced by a triple $\mathcal{I}, \tau, Q$ formed by an open interval $\mathcal{I}$, a variable $\tau \in \mathcal{I}$, and a positive C^∞ function Q of the variable τ. This triple is subject to just one condition, reflecting positivity of f, and stating that $\varepsilon(\tau - c) > 0$ for all $\tau \in \mathcal{I}$, unless $\varepsilon = 0$, or, equivalently, either $\varepsilon = 0$, or $c \notin \mathcal{I}$ and $\varepsilon = \mathrm{sgn}\,(\tau - c)$ whenever $\tau \in \mathcal{I}$.

In addition, Q, treated as a function $M' \to \mathbf{R}$, is given by $Q = g(\nabla\tau, \nabla\tau)$.

In fact, let us set $Q = a^2r^2\theta$. As $d\tau/dr = ar\theta \neq 0$ for all $r \in \mathcal{J}$, the assignment $r \mapsto \tau$ constitutes a C^∞ diffeomorphism of $\mathcal{J}$ onto some interval $\mathcal{I}$, allowing us to replace r by the new variable $\tau \in \mathcal{I}$ and view Q as a function of τ. Now $Q = g(\nabla\tau, \nabla\tau)$, since $\langle v, v\rangle = a^2r^2$ (Remark 3.2) and $v = \nabla\tau$ (see above).

Conversely, given $\mathcal{I}, Q, \tau$ with $\varepsilon(\tau - c) > 0$ for all $\tau \in \mathcal{I}$ unless $\varepsilon = 0$, we may choose a diffeomorphism $\tau \mapsto r$ of $\mathcal{I}$ onto some interval $\mathcal{J} \subset (0, \infty)$ such that $d[\log r]/d\tau = a/Q$, and treat Q (originally, a function of $\tau \in \mathcal{I}$), as well as τ itself, as functions of the new variable $r \in \mathcal{J}$. We then define $\theta, f : \mathcal{J} \to (0, \infty)$ by $\theta = Q/(ar)^2$ (so that $d\tau/dr = ar\theta$), and $f = 1$ (when $\varepsilon = 0$) or $f = 2\varepsilon(\tau - c)$ with our given constant c (when $\varepsilon = \pm 1$).

17. Linearity of invariant connections

Lemma 17.1. *Let there be given a holomorphic real vector field v on a complex manifold M of complex dimension $m \geq 1$, (cf. § 5), a point $x \in M$ with $v(x) \neq 0$, and a real constant $a \neq 0$. Then there exists a biholomorphic identification of a neighborhood U of x in M with an open set in $\mathcal{L} \smallsetminus N$, for some holomorphic line bundle $\mathcal{L}$ over a complex manifold N, which makes v and $u = Jv$ appear as the vector fields* (3.3)*. Note that $N \subset \mathcal{L}$ according to the convention* (3.2)*.*

In fact, using the flow of v and the holomorphic inverse mapping theorem we can find holomorphic local coordinates $y_1, \ldots, y_m$ for M at x for which v and u are the coordinate vector fields for the real coordinate directions of $\operatorname{Re} y_m$ and $\operatorname{Im} y_m$. The new coordinates $y_1, \ldots, y_{m-1}, z$ with $z = \exp a y_m$ represent points near x by pairs (y, z), where $y = (y_1, \ldots, y_{m-1})$. We have thus biholomorphically identified a neighborhood of x in M with $N \times D$ for some open sets $N \subset \mathbf{C}^{m-1}$ and $D \subset \mathbf{C} \smallsetminus \{0\}$. Treating $N \times D$ as an open subset of the product line bundle $\mathcal{L} = N \times \mathbf{C}$, we easily obtain (3.3) for our v, u (for instance, expressing their flows first in terms of (y, y_m), and then with the aid of (y, z)). ■

For vector spaces V, W with $\dim V = 1$, linearity of mappings $V \to W$ follows from their homogeneity, which is the underlying principle of the next (well-known) lemma.

Lemma 17.2. *Suppose that $\mathcal{L}$ is the total space, with* (3.2)*, of a C^∞ complex line bundle over a manifold N, and $M' \subset \mathcal{L}$ is an open connected set having a non-empty connected intersection with every fibre $\mathcal{L}_y$, $y \in N$. Also, let v, u be the vector fields* (3.3) *on M' with a fixed real constant $a \neq 0$. A C^∞ distribution $\mathcal{H}$ on M' is the restriction to M' of the horizontal distribution of a C^∞ linear connection in $\mathcal{L}$ admitting a parallel Hermitian fibre metric $\langle \, , \rangle$ if and only if*

(a) *$\mathcal{H}$ is invariant under the local flows of both v and u,*
(b) *$TM' = \mathcal{H} \oplus \mathcal{V}$ for the vertical distribution $\mathcal{V}$, (cf. § 3),*
(c) *the real span of $\mathcal{H}$ and u is an integrable distribution on $M' \smallsetminus N$.*

Proof. Necessity of (a)–(c) is clear: the flows of v and u consist of multiplications by complex scalars in $\mathcal{L}$ (which gives (a)), (b) is obvious, and, finally, the norm function r of $\langle \, , \rangle$ is constant along both $\mathcal{H}$ and u (which yields (c)).

Conversely, for $\mathcal{H}$ with (a)–(c) and any $x \in M'$, let us use a local trivialization of $\mathcal{L}$ to identify a neighborhood of x in M' with $N' \times D$ for some open sets $N' \subset N$ and $D \subset \mathbf{C}$, and let Γ be the complex-valued 1-form on N' assigning, to any vector $w \in T_y N$, $y \in N'$, the complex number $\Gamma(w)$ such that the vector $(w, 0) \in T_{(y,z)}(N' \times D)$, for some (or any) $z \in D \smallsetminus \{0\}$, has the $\mathcal{V}$ component $(0, \Gamma(w) z)$ (or, equivalently, its $\mathcal{H}$ component is $(w, -\Gamma(w) z)$). Note that Γ is well defined, that is, independent of the choice of z, since (a) implies invariance of $\mathcal{H}$, in addition to that of $\mathcal{V}$, under multiplications by complex scalars in $\mathcal{L}$.

As $(0, \zeta)^{\mathrm{hrz}} = 0$ whenever $(0, \zeta) \in T_{(y,z)}(N' \times D)$, the second formula in (3.5) now shows that $\mathcal{H}$ coincides, on $N' \times D$, with the horizontal distribution of the unique linear connection in $\mathcal{L}'$ (the restriction of $\mathcal{L}$ to N') whose connection 1-form in our fixed local trivialization (cf. Remark 3.1) is Γ. Since the intersections

$M' \cap \mathcal{L}_y$ are non-empty and connected, these 'local' connections fit together to form a single 'global' linear connection in $\mathcal{L}$ with the horizontal distribution $\mathcal{H}$.

Finally, by (c), locally in $M' \smallsetminus N$ there exist C^∞ functions φ with $d_v\varphi > 0$ and $d_u\varphi = d_w\varphi = 0$ for all w in $\mathcal{H}$. As $[v,u] = 0$, (a) implies constancy of $d_v\varphi$ along u and $\mathcal{H}$ as well. (For instance, $d_w d_v\varphi = 0$ for all C^∞ sections w of $\mathcal{H}$, since $d_v d_w\varphi = 0$ and $\mathbf{L}_v w = [v,w]$ is, by (a), a section of $\mathcal{H}$.) Thus, $d_v\varphi = \Upsilon(\varphi)$ for some positive C^∞ function Υ of a real variable. Choosing a positive function r of the variable φ with $\Upsilon(\varphi)\, d[\log r]/d\varphi = a$, and treating it as a function on an open subset of $M' \smallsetminus N$, we now get $d_v r = ar$ and $d_u r = 0$, so that, by (3.4), r is the restriction to an open subset of $\mathcal{L}$ of the norm function of some Hermitian fibre metric $\langle\,,\rangle$ in $\mathcal{L}$ (Remark 3.2). The linear connection that we found makes $\langle\,,\rangle$ parallel (that is, its parallel transport is norm-preserving), since r is constant along the horizontal distribution $\mathcal{H}$. This completes the proof. ■

Remark 17.3. Given a special Kähler-Ricci potential τ (see (7.1)) on a Kähler manifold (M,g) of complex dimension $m \geq 1$, a point $x \in M$ with $(d\tau)(x) \neq 0$, and a fixed real constant $a \neq 0$, let $v, u, \mathcal{V}, \mathcal{H}$ be as in (7.3). Then there exists a holomorphic line bundle $\mathcal{L}$ over a complex manifold N along with a biholomorphic identification of a neighborhood of x in M with an open subset of $\mathcal{L}\smallsetminus N$ (notation of (3.2)), under which v, u and $\mathcal{V}$ become the vector fields (3.3) and the vertical distribution in $\mathcal{L}$ (§ 3), while $\mathcal{H}$ appears as the horizontal distribution of some C^∞ linear connection in $\mathcal{L}$ that admits a parallel Hermitian fibre metric $\langle\,,\rangle$.

In fact, v is holomorphic (by (7.1) and Lemma 5.2(ii)). Applying Lemma 17.1 we may choose $N, \mathcal{L}$ and a neighborhood U of x in M, biholomorphically identified with an open set in $\mathcal{L} \smallsetminus N$ so that v, u are given by (3.3) (and $\mathcal{V} = \mathrm{Span}\,\{v,u\}$ is the vertical distribution). Our claim will now follow from Lemma 17.2, once we have shown that conditions (a)–(c) in Lemma 17.2 are satisfied. (The 'non-empty connected intersections' property of U will hold once we make U smaller and replace N by a suitable open submanifold.)

First, (b) is obvious. To establish (a), let us consider any vector field w commuting with v and u. Such fields w exist, locally, and realize all vectors at any given point x, as one verifies by prescribing w along a submanifold with the tangent space $\mathcal{H}_x$ at x, and then spreading it over a neighborhood of x using the flows of v and u (which commute, cf. (5.1.b)). Setting $\vartheta = g(v,w)$ and $\vartheta' = g(u,w)$, and using the Leibniz rule along with the equalities $\nabla_v w = \nabla_w v$, $\nabla_u w = \nabla_w u$ (see (2.1.ii)), the fact that ∇v is self-adjoint (as $v = \nabla\tau$, cf. end of § 2), while ∇u is skew-adjoint (as u is Killing; see (7.1) and Remark 5.1) and, finally, Lemma 7.5(ii), we obtain $d_v\vartheta = 2\psi\vartheta$, $d_v\vartheta' = 2\psi\vartheta'$, $d_u\vartheta = d_u\vartheta' = 0$. (For instance, $d_v\vartheta = g(\nabla_v v, w) + g(v, \nabla_v w)$, while $g(v, \nabla_v w) = g(v, \nabla_w v) = g(\nabla_v v, w) = \psi g(v,w) = \psi\vartheta$.) Thus, if $\vartheta = \vartheta' = 0$ at one point of a leaf of $\mathcal{V} = \mathrm{Span}\,\{v,u\}$, we have $\vartheta = \vartheta' = 0$ everywhere in the leaf (due to uniqueness of solutions for ordinary differential equations), which proves (a). Finally, integrability of $\mathrm{Span}_{\mathbf{R}}(\mathcal{H}, u)$ (that is, (c)) is clear as $\mathrm{Span}_{\mathbf{R}}(\mathcal{H}, u) = v^\perp$ and $v = \nabla\tau$ (by (7.5) and (7.3)).

Remark 17.4. Let $\mathcal{L}$ be a holomorphic line bundle over a complex manifold N. Every Hermitian fibre metric $\langle\,,\rangle$ in $\mathcal{L}$ then has a *Chern connection* ([**5, 14**]), that is, a unique linear connection in $\mathcal{L}$ making $\langle\,,\rangle$ parallel and having a J-invariant horizontal distribution, J being the complex structure tensor on the total space.

In fact, every real subspace V of real codimension one in a complex vector space T with $\dim T < \infty$ contains a unique complex subspace H of complex codimension one (in T), as one sees fixing a Hermitian inner product $(\,,)$ in T and a non-zero vector w which is $\mathrm{Re}(\,,)$-orthogonal to V, and noting that all complex subspaces $\mathrm{Re}(\,,)$-orthogonal to w are also $(\,,)$-orthogonal to w, so that, for dimensional reasons, our H must be the $(\,,)$-orthogonal complement of w.

Applying this, at any $x \in \mathcal{L} \smallsetminus N$ (cf. (3.2)), to $T = T_x\mathcal{L}$ and V which is the tangent space at x of the submanifold given by $r = r(x)$, where r is the norm function of $\langle\,,\rangle$ (Remark 3.2), we obtain a space $H = \mathcal{H}_x \subset V \subset T_x\mathcal{L}$. Along with $\mathcal{H}_y = T_yN$ for $y \in N \subset \mathcal{L}$, this defines the horizontal distribution $\mathcal{L} \ni x \mapsto \mathcal{H}_x$ of a linear connection making $\langle\,,\rangle$ parallel. In fact, assumptions (a)–(c) in Lemma 17.2 are satisfied: (a) due to uniqueness of each $H = \mathcal{H}_x$, and (c) since r is constant along $\mathrm{Span}_{\mathbf{R}}(\mathcal{H}, u)$.

18. A local classification theorem

In § 8 we constructed a family of special Kähler-Ricci potentials τ on Kähler manifolds (M, g) of any complex dimension $m \geq 1$. We now show that those examples are, locally, the only ones possible. Although we establish this here just at points in general position, a similar assertion holds at all points (see [**12**, § 16]).

Theorem 18.1. *Let τ be a special Kähler-Ricci potential, with* (7.1)*, on a Kähler manifold (M, g) of complex dimension $m \geq 1$, and let $M' \subset M$ be the open set where $d\tau \neq 0$. Then M' is dense in M and every point of M' has a neighborhood on which, up to a biholomorphic isometry, g and τ are obtained as in* § 8.

Proof. Denseness of M' was proved in Corollary 12.3. Applying Remark 17.3 to M, g, τ with any fixed constant $a \neq 0$ and a given point $x \in M'$, and then replacing M by a sufficiently small neighborhood of x, we may assume that $M = M'$ and identify M' biholomorphically with an open connected subset of $\mathcal{L} \smallsetminus N$, in such a way that this identification and $\mathcal{L}, N, v, u, \mathcal{V}, \mathcal{H}, \langle\,,\rangle$ have all the properties listed in Remark 17.3 plus the 'non-empty connected intersections' property of Lemma 17.2. (For the latter, it may be necessary to replace M', N by smaller open submanifolds.) Depending on whether $\phi = 0$ identically or $\phi \neq 0$ everywhere in M' (see Lemma 12.5), let us define $f : M' \to (0, \infty)$ by $f = 1$ or, respectively, $f = Q/|\phi|$ with $Q = g(\nabla\tau, \nabla\tau)$, and set, in both cases, $\varepsilon = \operatorname{sgn}\phi \in \{-1, 0, 1\}$, so that $f\phi = \varepsilon Q$. Thus, in the latter case, $f = 2|\tau - c| = 2\varepsilon(\tau - c)$, with c as in Lemma 12.5.

There exists a unique Riemannian metric h on N such that $g = f\pi^*h$ on $\mathcal{H}$, where $\pi : \mathcal{L} \to N$ is the bundle projection. In fact, as $\pi(M') = N$, we may define h by $h(w, w') = g(w, w')/f$, with w, w' standing for any two C^∞ vector fields on N, as well as for their horizontal lifts to $M' \subset \mathcal{L}$. Namely, Corollary 13.2 (when $\phi = 0$) or (13.2.ii) (when $\phi \neq 0$) shows that $g(w, w')/f : M' \to \mathbf{R}$ is constant in the vertical directions, and hence it is well-defined as (that is, descends to) a function $N \to \mathbf{R}$. (Since the flows of v, u in $\mathcal{L}$ consist of fibrewise multiplications by complex constants, the assumptions about w, w' in (13.2.ii) mean that they are the horizontal lifts of some vector fields in N.) Furthermore, denoting by r the norm function of $\langle\,,\rangle$, we have $\langle v,v\rangle = \langle u,u\rangle = a^2r^2$ and $\mathrm{Re}\langle v,u\rangle = 0$ (see Remark 3.2), and so $g = \theta\,\mathrm{Re}\langle\,,\rangle$ on $\mathcal{V}$, with $\theta = Q/(ar)^2$. Since $g(\mathcal{H},\mathcal{V}) = \{0\}$ (that is, $\mathcal{H} = \mathcal{V}^\perp$, cf. (7.3)), our g thus is given by the same formulae as in § 8, and all we need to verify is that the data (8.1) just obtained (with $\mathcal{J}$ standing for the range of $r : M' \to \mathbf{R}$) have all the properties required in § 8.

First, making M', N even smaller if necessary, we may assume that τ is a function of $r : M' \to \mathbf{R}$ (as $v = \nabla\tau$ and ∇r are both g-orthogonal to u and $\mathcal{H}$, cf. (7.5), (7.3), (3.4)); hence so are Q (see Lemma 11.1(a)) and θ, f. Also, $d\tau/dr = d_v\tau/d_vr = Q/(ar) = ar\theta$ by (2.4.i) and (3.4).

Next, $\pi : M' \to N$ is holomorphic and $\mathcal{H}$ is J-invariant (since so is $\mathcal{V}$, cf. (7.3)). Thus, if we use the same symbol J for the complex structure tensors of both M' and N, no ambiguity will arise as to the meaning of Jw, where vectors w tangent to N are identified with their horizontal lifts to $M' \subset \mathcal{L}$. The metric h on N now is clearly Hermitian, since so is g, on M'. Applying Lemma 15.1 to $\pi : M' \to N$, we see that, for w as above, D_w commutes with J in N, as ∇_w and $...^{\mathrm{hrz}}$ both commute with J in M' (due to the fact that g is Kähler, and J-invariance of $\mathcal{H}$). Consequently, J is D-parallel in N, that is, h is a Kähler metric on the complex manifold N. Moreover, $\pi : M' \to N$ and our $f, \varepsilon, v = \nabla\tau$, along with ϕ, ψ as in (7.4), satisfy the hypotheses of Corollary 15.3 (provided that $m \geq 2$): (15.1) is our definition of h (with $[\nabla f]^{\mathrm{hrz}} = 0$ as $v = \nabla\tau$ is vertical, cf. (7.3), and f is a linear function of τ), while (15.3) and (14.3.c) (for $\beta = -2\varepsilon\omega/f$, with f, ε as above) follow from Lemma 7.5(ii), (13.2.i) and (13.1.b). Relation $\mathrm{r} = \lambda g$ on $\mathcal{H}$ in (7.4) now yields the premise of Corollary 15.3(ii), so that, by Corollary 15.3, (N, h) is an Einstein manifold unless $m = 2$. This is also trivially true when $m = 1$, that is, N consists of a single point.

Finally, as $g = f\pi^*h$ on $\mathcal{H}$, (4.1) gives $\omega = f\pi^*\omega^{(h)}$ on $\mathcal{H}$ for the Kähler forms $\omega^{(h)}$ and ω of h and g. Since $f\phi = \varepsilon Q$ (see above), combining (3.6), (3.3) and (13.1.b) we conclude that our connection in $\mathcal{L}$ has the curvature form $\Omega = -2\varepsilon a\,\omega^{(h)}$. This completes the proof. ■

19. Solutions to (10.1)–(10.3) with $\phi = 0$

Given an integer $m \geq 2$, a septuple $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ of C^∞ functions of a variable τ satisfies conditions (10.1)–(10.3) on some interval on which $\phi = 0$ identically if and only if, for some constants K, α, η with $|K| + |\alpha| + |\eta| > 0$,

$$Q = -K\tau^2 + (2m-1)^{-1}[\alpha\tau^{2m-1} - \eta/m]\,, \tag{19.1}$$

while $Y = -2K\tau + \alpha\tau^{2m-2}$, $\mathrm{s} = -(2m-1)(2m-4)K - 2(m-1)\alpha\tau^{2m-3}$, $\psi = -K\tau + \alpha\tau^{2m-2}/2$, $\lambda = (3-2m)K$, $\mu = K - (m-1)\alpha\tau^{2m-3}$ and, of course, $\phi = 0$. Note that positivity of $|K| + |\alpha| + |\eta|$ amounts to (10.3.i) on *some* interval.

In fact, a septuple just defined is easily seen to satisfy (10.1)–(10.3) on a suitable interval. Conversely, let us assume that (10.1)–(10.3) hold with $\phi = 0$ on an interval $\mathcal{I}$. By (10.1), λ then is constant, that is, $\lambda = (3-2m)K$ for some $K \in \mathbf{R}$. If $\psi = \phi = 0$ everywhere in $\mathcal{I}$, (10.3.iii) gives $\lambda = \mu$, while $Y = 0$ and $\mu = 0$ identically by (10.3.ii) and (10.1), that is (cf. (10.1), (10.3.ii)), Q is constant and $Y = \mathrm{s} = \phi = \psi = \lambda = \mu = 0$ on $\mathcal{I}$. (This is a special case of the above formulae, with $K = \alpha = 0$, $\eta \neq 0$.) On the other hand, if $\psi \neq \phi$ on a dense subset of $\mathcal{I}$, (10.3.ii) gives $2(2m-3)\mu = (2m-3)[\mathrm{s} + (2-2m)\lambda]$ and, by (10.2.ii), (10.3.iii), $\tau d\mathrm{s}/d\tau = (2m-3)\mathrm{s} - (2m-1)(2m-4)\lambda$. Replacing λ in this equality with $(3-2m)K$ and multiplying by τ^{2-2m}, we obtain the required formula for s, with α representing a constant of integration. As $\psi - \phi = \psi = Y/2$ (due to (10.3.ii) with $\phi = 0$) and $2(\lambda - \mu) = 2m\lambda - \mathrm{s}$ (by (10.3.ii)), with $\lambda = (3-2m)K$, we may rewrite (10.3.iii) as $(m-1)Y = [m(3-2m)K - \mathrm{s}/2]\tau$ which, combined with the formula for s, gives the expressions for Y and ψ. The first two equations in (10.2) now yield the formula for μ and (19.1) for some $\eta \in \mathbf{R}$.

Finally, let $\psi \neq \phi$ somewhere in $\mathcal{I}$, and let $\mathcal{I}'$ be a maximal subinterval such that $\psi \neq \phi$ on a dense subset of $\mathcal{I}'$. If one of the endpoints of $\mathcal{I}'$ were an interior point of $\mathcal{I}$, our discussion of the case $\psi = \phi$ would imply vanishing, at that point, of λ and all derivatives of μ, so that on $\mathcal{I}'$ we would have the above formulae with $K = \alpha = 0$, contradicting the assumption that $\psi \neq \phi = 0$.

20. A single equation, equivalent to (10.1)–(10.4)

As in Lemma 11.1, we will refer to individual equations in the multi-formula expressions (10.1) and (20.1) by labelling them with lower-case Roman numerals: (10.1.i)–(10.1.v), (20.1.i)–(20.1.v).

For a fixed integer $m \geq 1$ and any solution $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ to (10.1)–(10.4),

$$\begin{aligned} &Q = 2(\tau - c)\phi, \quad \psi = \phi + (\tau - c)\phi', \quad Y = 2m\phi + 2(\tau - c)\phi', \\ &\mu = -(m+1)\phi' - (\tau - c)\phi'', \qquad \lambda - \mu = 2(m-1)(\tau - c)\phi'/\tau, \end{aligned} \tag{20.1}$$

where c is the constant in Lemma 10.1 and $\phi' = d\phi/d\tau$. In fact, (20.1.i) is obvious; differentiating (20.1.i) and using (10.1.i), we obtain (20.1.ii); (20.1.iii) is clear from (10.3.ii) and (20.1.ii); differentiating (20.1.iii) we get (20.1.iv) (cf. (10.1.ii)); while (10.3.iii) gives (20.1.v), as $\psi - \phi = (\tau - c)\phi'$ by (20.1.ii).

Remark 20.1. A solution $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ to (10.1)–(10.4) with a fixed integer $m \geq 1$ is uniquely determined by its constituent function ϕ and the constant c defined in Lemma 10.1. Explicitly, formulae for Q, Y, ψ, λ, μ (or, s) in terms of ϕ and c are provided by (20.1) (or, by (10.3.ii) and (20.1.iv)–(20.1.v)).

Lemma 20.2. *Given an integer $m \geq 2$ and a C^1 solution $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ to* (10.1)–(10.4) *on an interval, let c be the constant in* Lemma 10.1*. Then*

$$\tau^2(\tau - c)\frac{d^3\phi}{d\tau^3} = [(m-4)\tau^2 - 2(m-1)c\tau]\frac{d^2\phi}{d\tau^2} + 2(m-1)(\tau + c)\frac{d\phi}{d\tau}. \tag{20.2}$$

Proof. We have (20.2) on every subinterval where $\phi' = d\phi/d\tau$ vanishes identically, and so we may restrict our consideration to a fixed subinterval in which $\phi' \neq 0$ everywhere and $0 \neq \tau \neq c$. (The union of all subintervals of one or the other type is dense in the original interval of the variable τ.) By (20.1.ii), $\psi - \phi = (\tau - c)\phi'$, and so, from (20.1.i), $(\psi - \phi)\phi = Q\phi'/2$. Thus, if we replace $2(m-1)(\psi - \phi)$ on the left-hand side of (10.2.i) with $(\lambda - \mu)\tau$ (cf. (10.3.iii)) and then divide by $(\lambda - \mu)Q$, we obtain $\tau\mu' = \lambda - \mu + 2(m-1)\mu + 2(m-1)^2\phi'$. (Note that $\lambda \neq \mu$

everywhere, due to our choice of the subinterval and (20.1.v).) Replacing $\lambda - \mu$ and then $2(m-1)\mu$ with the expressions provided by (20.1.iv)–(20.1.v), we now have $\tau\mu' = -4(m-1)\phi' - 2(m-1)(\tau-c)\phi'' + 2(m-1)(\tau-c)\phi'/\tau$. On the other hand, differentiating (20.1.iv) we get $\tau\mu' = -(m+2)\tau\phi'' - (\tau-c)\tau\phi'''$. Equating the last two formulae for $\tau\mu'$ we arrive at (20.2), which completes the proof. ■

Proposition 20.3. *Given an integer $m \ge 2$, a real number c and an interval of the variable τ, not containing 0 or c, there exists a bijective correspondence between C^1 solutions $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ to* (10.1)– (10.4) *defined on our interval, for which c is the constant in* Lemma 10.1*, and C^3 functions ϕ on this interval which are non-zero everywhere and satisfy equation* (20.2)*. The correspondence in question associates with $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ the constituent function ϕ.*

Proof. Any solution $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ to (10.1)–(10.4) has $\phi \ne 0$ everywhere (cf. (10.4)) and the assignment $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu) \mapsto \phi$ is injective (Remark 20.1). It is also surjective since, whenever (20.2) and (10.4) hold, the functions $Q, Y, \phi, \psi, \lambda, \mu$ and s, determined by ϕ as in Remark 20.1, must satisfy (10.1)–(10.3). In fact, (10.1.i) follows from (20.1.i) and (20.1.ii); (10.1.ii) from (20.1.iii) and (20.1.iv); (10.1.iv) from (20.1.ii), (20.1.iv), (20.1.i) and, again, (20.1.ii); while (10.1.iii) is clear since (20.1.i) and (20.1.ii) give $Q\phi' = 2(\tau-c)\phi\phi' = 2(\psi-\phi)\phi$. Also, writing both $\lambda' - \mu'$ and μ' in terms of ϕ and its derivatives (via (20.1.iv), (20.1.v)), then replacing ϕ''' with the expression provided by (20.2), and noting that the occurrences of ϕ'' in the resulting formula for $\lambda' = (\lambda' - \mu') + \mu'$ cancel one another, we easily obtain (10.1.v) from (20.1.i) and (20.1.v). Next, (10.3.i) follows from our assumptions $\phi \ne 0$ and $\tau \ne c$, combined with (20.1.i); the first formula in (10.3.ii) is immediate from (20.1.iii) and (20.1.ii), while the second one is the equality defining s in Remark 20.1; and (10.3.iii) is clear from (20.1.ii) and (20.1.v). As (10.1) and (10.3) now imply (10.2) (Remark 10.2), this completes the proof. ■

Remark 20.4. A C^3 function ϕ of the variable τ, defined on an interval $\mathcal{I}$ such that $0 \notin \mathcal{I}$, satisfies (20.2) (with a constant c and a fixed integer $m \ge 2$) if and only if the function

$$m\phi - (\tau-c)^2\phi'' - m(\tau-c)\phi' + 2(m-1)(\tau-c)^2\phi'/\tau, \tag{20.3}$$

with $\phi' = d\phi/d\tau$, is constant on $\mathcal{I}$. (To see this, just differentiate (20.3).)

Remark 20.5. Proposition 20.3 and Remark 20.1 lead to an explicit description of those solutions $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ to (10.1)–(10.4) for which c defined in Lemma 10.1 is zero. Namely, for a fixed integer $m \ge 2$, (20.2) with $c = 0$ is an Euler equation, and its solution space has a basis formed by the power functions 1, τ^m and $1/\tau$. As $Q = 2\phi\tau$ (cf. (20.1)), there exist constants K, α, η with

$$(20.4)\qquad \begin{aligned} &\text{i)}\quad 2m\phi = K + m\alpha\tau^m - 2(m+1)^{-1}\eta/\tau,\\ &\text{ii)}\quad Q = m^{-1}K\tau + \alpha\tau^{m+1} - 2(m+1)^{-1}\eta/m. \end{aligned}$$

21. Solutions with $c \neq 0$

Lemma 21.1. *For any fixed integer* $m \geq 1$*, the rational functions*

$$F(t) = \frac{(t-2)t^{2m-1}}{(t-1)^m}, \qquad E(t) = (t-1)\sum_{k=1}^{m} \frac{k}{m}\binom{2m-k-1}{m-1} t^{k-1} \tag{21.1}$$

of the real variable t *and their derivatives* $\dot{F}, \dot{E}$ *satisfy the equations*

$$\begin{aligned} &\text{a)} \quad t(t-2)\dot{F}(t) = \Xi(t)F(t), \\ &\text{b)} \quad t(t-2)\dot{E}(t) = \Xi(t)E(t) + 2(2m-1)E(0), \quad \text{where} \\ &\text{c)} \quad \Xi(t) = m(t-1) + m(t-1)^{-1} - 2(m-1). \end{aligned} \tag{21.2}$$

Proof. Relation (21.2.a) is immediate from (21.1). Using a subscript to mark the dependence of F, E on m, we have $(t-1)[E_m(t) - E_m(0)] = t^2 E_{m-1}(t)$ for $m \geq 2$ (as one sees expanding the difference into powers of t via (21.1)), while (21.1) gives $(t-1)F_m(t) = t^2 F_{m-1}(t)$ and $mE_m(0) = 2(2m-3)E_{m-1}(0)$. Also,

$$\begin{aligned} &\text{i)} \quad [E_m - E_m(0)]/F_m = E_{m-1}/F_{m-1} \qquad \text{for} \quad m \geq 2, \\ &\text{ii)} \quad t(t-2)\, d[E(t)/F(t)]/dt = 2(2m-1)E(0)/F(t) \qquad \text{for} \quad m \geq 1. \end{aligned} \tag{21.3}$$

In fact, (21.3.i) is obvious from the preceding equalities, while (21.3.ii), easily verified for $m = 1$, can be established by induction: let (21.3.ii) hold when some given $m \geq 2$ is replaced by $m-1$. The above recursion formulae for $F(t)$ and $E(0)$ now yield $t(t-2)\, d[E_{m-1}/F_{m-1}]/dt = mt^2 E_m(0)/[(t-1)F_m]$. Using this to evaluate $t(t-2)\, d[E_m/F_m]/dt$ from (21.3.i) (where $d[E_m(0)/F_m]/dt$ is computed with the aid of (21.2.a)), and noting that $mt^2(t-1)^{-1} - \Xi(t) = 2(2m-1)$ for Ξ as in (21.2.c), we obtain (21.3.ii) for the given m, as required. Now, as $E = (E/F)F$, (21.2.b) follows from (21.3.ii) and (21.2.a), which completes the proof. ∎

Note that applying d/dt to (21.2.a) and (21.2.b) we obtain

$$t(t-2)\ddot{\phi} + 2(t-1)\dot{\phi} = \dot{\Xi}\phi + \Xi\dot{\phi}, \quad \text{both for} \quad \phi = E \quad \text{and} \quad \phi = F, \tag{21.4}$$

where $\dot{\phi} = d\phi/dt$ and $\phi, \dot{\phi}, \Xi, \dot{\Xi}$ stand for $\phi(t), \dot{\phi}(t), \Xi(t), \dot{\Xi}(t)$.

Lemma 21.2. *Given an integer* $m \geq 1$*, a constant* $c \in \mathbf{R} \smallsetminus \{0\}$*, and an interval* $\mathcal{I}$ *of the variable* τ*, not containing* 0 *or* c*, the vector space* V *of all* C^3 *solutions* ϕ *to the linear equation* (20.2) *on* $\mathcal{I}$ *has a basis consisting of the constant function* 1 *along with* $E(t)$ *and* $F(t)$*, defined by* (21.1)*, and treated as functions of the variable* τ *via the substitution* $t = \tau/c$*.*

In fact, d/dt applied to (21.4) yields a third-order linear differential equation imposed on ϕ. Using (21.4) again to express ϕ through $\dot{\phi}$ and $\ddot{\phi}$, and substituting that expression for ϕ (but not its derivatives) in our third-order equation, we obtain (20.2) with τ and c replaced everywhere by t and 1. (Note that $\dot{\Xi} = m - m/(t-1)^2 = mt(t-2)/(t-1)^2$, $\ddot{\Xi} = 2m/(t-1)^3$, and $\ddot{\Xi}/\dot{\Xi} = 2/[t(t-1)(t-2)]$.)

However, this modified version of (20.2), satisfied by both $\phi = E$ and $\phi = F$, is equivalent to (20.2) under the variable change $\tau = ct$. ■

Remark 21.3. Given an integer $m \geq 2$, let $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ be a C^1 solution to (10.1)– (10.4) on some given interval, for which c in Lemma 10.1 is non-zero. By Lemmas 20.2 and 21.2, the constituent function ϕ is a linear combination of $E(t)$, $F(t)$ and 1, treated as functions of $\tau = ct$. Since Q is related to ϕ as in (20.1), there exist constants A, B, C such that $|A| + |B| + |C| > 0$ (cf. (10.3.i)) and

$$(21.5) \quad Q = (t-1)[A + BE(t) + CF(t)] \text{ with } t = \tau/c \text{ and } F, E \text{ as in (21.1).}$$

(About notations, see Remark 21.4 below.) Conversely, let Q be a function of the variable τ given by (21.5) with constants A, B, C, c such that $c \neq 0$ and $|A| + |B| + |C| > 0$. Then Q, restricted to any interval $\mathcal{I}$ on which $(\tau - c)\tau Q \neq 0$, arises in the manner just described from a unique C^1 solution to (10.1)–(10.4) for which c is the constant in Lemma 10.1; explicitly, $Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu$ are obtained as in Remark 20.1. In fact, by Lemma 21.2, $\phi = Q/[2(\tau - c)]$ then satisfies (20.2), and so such a solution exists and is unique in view of Proposition 20.3.

As $\phi = Q/[2(\tau - c)]$ is a solution to (20.2) on $\mathcal{I}$ and $0 \notin \mathcal{I}$, the expression (20.3) is constant on $\mathcal{I}$ and has a simple expression in terms of A, B, C, c in (21.5). Specifically, (20.3) equals $mA/(2c)$. In fact, according to Remark 20.4, by assigning (20.3) to ϕ we define a real-valued linear functional on the three-dimensional space V of solutions mentioned in Lemma 21.2. Similarly, $mA/(2c)$ is such a functional, assigning to ϕ the number $m/(2c)$ times the coefficient A in the expansion (21.5) of the function Q related to ϕ via the first formula in (20.1). That these two functionals coincide is now clear since they agree on the basis $1, E, F$ of the solution space: namely, the former functional yields m for $\phi = 1$ (which is obvious) and 0 for $\phi = E$ or $\phi = F$ (which is nothing other than (21.4), with $\dot{\Xi} = mt(t-2)/(t-1)^2$, rewritten in terms of the variable $\tau = ct$).

Remark 21.4. Having run out of letters, we had to choose between using unorthodox notation and allowing different meanings of identical or similar symbols. We decided to do the latter; as a result, B in (21.5) is not the same as in (2.6), C in (21.5) is to be distinguished from the lower-case c, and the italic r, s for the norm function and various curve parameters must not be confused with the roman r, s for the Ricci tensor and scalar curvature.

22. Three types of conformally-Einstein Kähler manifolds

Proposition 22.1. *Let M, g, m, τ satisfy* (0.1) *with $m \geq 3$, or* (0.2) *with $m = 2$, and let $Q : M \to \mathbf{R}$ be given by $Q = g(\nabla\tau, \nabla\tau)$. Then Q is a rational function of τ. More precisely, the open set $M' \subset M$ on which $d\tau \neq 0$ is connected and dense in M and, for ϕ, c as in* Lemma 12.5*, one of the following three cases occurs*:

(a) $\phi = 0$ *identically on* M'.
(b) $\phi \neq 0$ *everywhere in* M' *and* $c = 0$.
(c) $\phi \neq 0$ *everywhere in* M' *and* $c \neq 0$.

In case (a)*,* (b)*, or* (c)*, the functions* $\tau, Q : M \to \mathbf{R}$ *satisfy* (19.1)*, or* (20.4.ii) *or, respectively,* (21.5)*, for some constants* K, α, η*, or* A, B, C*. In case* (c)*,* $\tau \neq c$ *everywhere in* M *unless* $C = 0$ *in* (21.5)*.*

Proof. Corollary 9.3 yields (7.1), so that M' is connected and dense (Corollary 12.3) and, by Lemma 12.5, we have (a), (b), or (c). However, Corollary 11.2 gives (10.1)–(10.3), everywhere in M', for the septuple $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ associated with τ. Therefore, in case (a) (or (b), or (c)), any given point of M' has a neighborhood in which (19.1) (or (20.4.ii), or (21.5)) holds for some constants K, α, η or A, B, C. (See § 19 and Remarks 20.5, 21.3.) Consequently, this triple of constants forms a locally constant function $M' \to \mathbf{R}^3$ which (as M' is connected) must be constant. Hence (19.1), or (20.4.ii), or (21.5) holds *everywhere* in M', with a single triple of constants. Denseness of M' in M implies the same everywhere in M, while (21.5) yields $\tau \neq c$ on M unless $C = 0$, since Q given by (21.5) with $C \neq 0$ has a pole at $t = 1$. This completes the proof. ∎

23. Examples of conformally-Einstein Kähler metrics

Suppose that $m \geq 2$ is an integer, $\mathcal{I} \subset \mathbf{R}$ is an open interval of the variable τ, while Q is a rational function of τ, analytic and positive everywhere in $\mathcal{I}$, and

(i) Q is defined by (19.1) for some constants K, α, η, or
(ii) Q has the form (20.4.ii) with some constants K, α, η, or, finally,
(iii) Q is given by (21.5) for some constants A, B, C, c with $c \neq 0$.

We further set $c = 0$ in case (ii); in case (i), we leave c undefined. Replacing $\mathcal{I}$ with a subinterval, we also assume that $c \notin \mathcal{I}$ in cases (ii), (iii), and define $\varepsilon \in \{-1, 0, 1\}$ by $\varepsilon = 0$ (case (i)) or $\varepsilon = \operatorname{sgn}(\tau - c)$ for all $\tau \in \mathcal{I}$ (cases (ii), (iii)).

Using our Q and a fixed real constant $a \neq 0$, let us now select a positive C^∞ function r on $\mathcal{I}$ with $Q\,dr/d\tau = ar$. Thus, $\tau \mapsto r$ is a diffeomorphism of $\mathcal{I}$ onto an open interval $\mathcal{J} \subset (0, \infty)$. Replacing τ with r, we will treat functions of $\tau \in \mathcal{I}$, including Q, as functions of $r \in \mathcal{J}$. Further C^∞ functions of $r \in \mathcal{J}$, introduced in this way, are $\theta = Q/(ar)^2$ and f, with $f = 1$ in case (i) and $f = 2|\tau - c| = 2\varepsilon(\tau - c)$ in cases (ii), (iii).

Let (N, h) be a Kähler-Einstein manifold of complex dimension $m - 1$ with the Ricci tensor $\mathrm{r}^{(h)} = \kappa h$, where the constant κ is defined by

$$\text{(23.1)} \quad \kappa = (3 - 2m)K \text{ in case (i)}, \; \kappa = \varepsilon K \text{ in case (ii)}, \; \kappa = \varepsilon m A/c \text{ in case (iii)}.$$

Also, let there be given a holomorphic line bundle $\mathcal{L}$ over N with a Hermitian fibre metric $\langle\,,\rangle$, and a C^∞ connection in $\mathcal{L}$ with a J-invariant horizontal distribution $\mathcal{H}$ (cf. Remark 17.4), making $\langle\,,\rangle$ parallel and having the curvature form (Remark 3.1) equal to $-2\varepsilon a$ times the Kähler form of (N, h) (see (4.1)). Using the symbol r also for the norm function of $\langle\,,\rangle$ (Remark 3.2), we will treat f, θ, τ, Q as functions $M' \to \mathbf{R}$, where $M' \subset \mathcal{L} \smallsetminus N$ is a fixed connected open subset of the preimage $r^{-1}(\mathcal{J})$ (notation of (3.2)). We now define a metric g on M' by declaring $\mathcal{H}$ to be g-orthogonal to the vertical distribution $\mathcal{V}$ (§ 3) and setting $g = f\pi^*h$ on $\mathcal{H}$ and $g = \theta\,\mathrm{Re}\langle\,,\rangle$ on $\mathcal{V}$. Here $\pi : \mathcal{L} \to N$ is the bundle projection, and $\mathrm{Re}\langle\,,\rangle$ is the standard Euclidean metric on each fibre of $\mathcal{L}$.

Remark 23.1. The above construction is, obviously, a special case of that in § 8, with a description modified as in Remark 16.1. Thus (see § 16), (M', g) is a Kähler manifold and τ is a special Kähler-Ricci potential on (M', g). There are *just three additional assumptions* on the data $\mathcal{I}, \tau, Q, a, \varepsilon, \ldots$ that distinguish the present case from the general situation in § 8: first, $m \geq 2$ (while, in § 8, $m = 1$ is also allowed); next, Q is required to satisfy (i), (ii) or (iii) above (rather than being just any positive C^∞ function of the variable $\tau \in \mathcal{I}$); and, finally, the function $\kappa : N \to \mathbf{R}$ with $\mathrm{r}^{(h)} = \kappa h$ is now assumed to be a specific constant with (23.1): in § 8 it need not be constant when $m = 2$, and can be *any* constant if $m \geq 3$.

Remark 23.2. If we relax the third 'additional assumption' in Remark 23.1 and just require κ with $\mathrm{r}^{(h)} = \kappa h$ to be a function on N, constant unless $m = 2$, our τ will still be a special Kähler-Ricci potential on the Kähler manifold (M', g) (cf. § 16). Condition (23.1) then holds if and only if the septuple $(Q, Y, \mathrm{s}, \phi, \psi, \lambda, \mu)$ associated with τ (Definition 7.2) satisfies (10.3.iii).

In fact, by (c) in § 16, $\phi = 0$ identically in case (i) (as $\varepsilon = 0$), while, in cases (ii), (iii), $\varepsilon = \pm 1$, and so $2\phi = Q/(\tau - c)$, with the constant c chosen above. Since, in all cases, $2\psi = dQ/d\tau$, $Y = 2\psi + 2(m-1)\phi$ and $2\mu = -dY/d\tau$ (Lemma 7.5), ψ, Y, μ are given either by the formulae following (19.1) (in case (i)), or by those listed in (20.1), with $\phi' = d\phi/d\tau$ (cases (ii), (iii)). Also, (b) in § 16 provides a formula for λ, with f as above. One now easily verifies that the difference $2(m-1)(\psi - \phi) - (\lambda - \mu)\tau$ (cf. (10.3.iii)) equals $[(3 - 2m)K - \kappa]\tau$ (in case (i), proving our claim), or $\tau/(\tau - c)$ times the difference between (20.3) and $\varepsilon\kappa/2$ (in cases (ii), (iii)). Our claim now also follows in case (ii), since (20.3) with $c = 0$ equals $K/2$ (by (20.4.i)). Finally, in case (iii), (20.3) equals $mA/(2c)$ (see Remark 21.3), so that (10.3.iii) is equivalent to $\kappa = \varepsilon mA/c$, as required.

Proposition 23.3. *Let (M', g) and $\tau, Q : M' \to \mathbf{R}$ be obtained as above, for some integer $m \geq 2$. Then (M', g) is a Kähler manifold and the conformally related metric $\tilde{g} = g/\tau^2$, defined wherever $\tau \neq 0$, is Einstein, while $Q = g(\nabla\tau, \nabla\tau)$ and the Laplacian of τ is a function of τ.*

In fact, according to Remark 23.1, (M', g) is a Kähler manifold and τ is a special Kähler-Ricci potential on (M', g) (as in (7.1)), with $g(\nabla\tau, \nabla\tau) = Q$ (see Remark 16.1). Since we assumed (23.1), Remark 23.2 gives (10.3.iii), and our assertion follows from Remark 9.4, the claim about the Laplacian $Y = \Delta\tau$ being obvious since ψ and ϕ are functions of τ and $Y = 2\psi + 2(m-1)\phi$ (see Remark 23.2). ■

24. Local structure of conformally-Einstein Kähler metrics

We can now prove our main local result: locally, at points in general position, and up to biholomorphic isometries, the only conformally-Einstein Kähler metrics in complex dimensions $m > 2$ are those described in Proposition 23.3.

Theorem 24.1. *Let $\tau : M \to \mathbf{R}$ be a non-constant C^∞ function on a Kähler manifold (M, g) of complex dimension $m \geq 2$ such that the conformally related metric $\tilde{g} = g/\tau^2$, defined wherever $\tau \neq 0$, is Einstein. If $m = 2$, let us also assume that $d\tau \wedge d\Delta\tau = 0$ everywhere in M. Then the open set $M' \subset M$ given* *by* $d\tau \ne 0$ *is dense in* M, *and every point of* M' *has a neighborhood on which* g *and* τ *are, up to a biholomorphic isometry, obtained as in* § 23 *with a rational function* Q *of the form* (19.1), *or* (20.4.ii), *or* (21.5).

More precisely, in cases (a), (b), (c) *of* Proposition 22.1, *the function* Q *used in the construction of* g *and* τ *satisfies, respectively,* (i), (ii), *or* (iii) *in* § 23. *In all cases* Q *equals* $g(\nabla\tau,\nabla\tau)$ *if one uses its dependence on* τ *to treat it as a function* $M \to \mathbf{R}$.

In fact, Corollary 9.3 yields (7.1), so that, by Theorem 18.1, M' is dense in M and, locally in M', our g,τ are obtained as in § 8. Moreover, the objects (8.1) used in § 8 may be replaced by the data $\mathcal{I},\tau,Q,a,\varepsilon,\dots$, with $Q = g(\nabla\tau,\nabla\tau) : M \to \mathbf{R}$ treated as a function of the variable $\tau \in \mathcal{I}$ (see Remark 16.1). All we now need to show is that the latter data satisfy the 'three additional assumptions' in Remark 23.1. To this end, note that $m \ge 2$, while Q has the form (i), (ii) or (iii) in § 23 (by Proposition 22.1). Finally, $\kappa : N \to \mathbf{R}$ with $\mathrm{r}^{(h)} = \kappa h$ is given by (23.1) according to Remark 23.2, as (10.3.iii) holds according to Remark 9.4. ■

Remark 24.2. For any given integer $m \ge 2$, the local biholomorphic-isometry types of quadruples (M,g,m,τ) satisfying (0.1) (when $m \ge 3$), or (0.2) (when $m = 2$), depend on three real constants along with an arbitrary local biholomorphic-isometry type of a Kähler-Einstein metric h in complex dimension $m-1$. In fact, Theorem 24.1 states that locally, at points with $d\tau \ne 0$, some data analogous to (8.1) with $\mathcal{J},r,\theta,\tau,f$ replaced by $\mathcal{I},\tau,Q$ lead to g and τ via an explicit construction. Of these data, a is not an invariant (as it may be arbitrarily chosen for any given g,τ, cf. Remark 17.3), while $\varepsilon \in \{-1,0,1\}$ is a discrete parameter and m is fixed, which leaves, aside from h, just the parameters A,B,C,c or K,α,η, representing Q in (i)–(iii) of § 23. Formula (23.1) for the Einstein constant κ of h now reduces the number of independent parameters by one, and, according to Remark 23.1, this is the only constraint (cf. Remark 24.3 below).

The parameters A,B,C,c or K,α,η explicitly appear in the geometry of (M,g) and τ, namely, through a functional relation between τ and $Q = g(\nabla\tau,\nabla\tau)$ (Proposition 22.1). Similarly, h may be explicitly constructed as a metric on a local leaf space for the distribution $\mathcal{V}$ with (7.3) (see the proof of Theorem 18.1).

The concept of a local-isometry type in Remark 24.2 is the one traditionally used in local differential geometry, where the restrictions of a real-analytic metric to two disjoint open subsets of the underlying manifold are regarded as belonging to the same type. The number of independent real parameters would, however, increase by 1 (for $m = 2$) or by $2m-1$ (for $m \ge 3$) if, instead, our 'types' involved a base point (at which $d\tau \ne 0$) and base-point preserving isometries.

Remark 24.3. Neither the fibre metric $\langle\,,\rangle$ nor the connection in $\mathcal{L}$, subject to the conditions listed in § 23, give rise to any additional parameters in Remark 24.2: locally, they form a single equivalence class. In fact, let us fix a holomorphic local trivializing section w of $\mathcal{L}$, defined on a contractible open set $N' \subset N$, and use this fixed w to represent connections in $\mathcal{L}$ through complex-valued 1-forms $\varGamma$, as in Remark 3.1. The conditions listed in § 23: J-invariance of the horizontal distribution $\mathcal{H}$, and a specific choice of the curvature form Ω, now mean that Γ is of type $(1,0)$ (that is, $\Gamma(y) : T_yN \to \mathbf{C}$ is complex-linear for every $y \in N'$, cf. (3.5)), and $d\Gamma = -i\Omega$ for a prescribed real-valued closed 2-form Ω of type $(1,1)$.

That a $(1,0)$ form Γ with $d\Gamma = -i\Omega$ exists on N' is clear as we may choose a function $\varphi : N' \to \mathbf{R}$ with $i\Omega = \partial\overline{\partial}\varphi$, and set $\Gamma = \partial\varphi$. Any other such form $\tilde{\Gamma}$ obviously equals $\Gamma - d\Phi$, where $\Phi : N' \to \mathbf{C}$ is holomorphic. The connections represented by Γ and $\tilde{\Gamma}$ thus are holomorphically equivalent, as the latter is the image of the former under the bundle automorphism (over N'), sending the section w (chosen above) to $\tilde{w} = e^{\Phi}w$. Finally, a connection with the curvature form Ω admits a parallel fibre metric $\langle\,,\rangle$, unique up to a constant factor, since $\langle w, w\rangle$ is a positive C^∞ function subject to the sole requirement that $d\log\langle w, w\rangle = 2\,\mathrm{Re}\,\Gamma$, the form $\mathrm{Re}\,\Gamma$ being closed due to real-valuedness of $i\,d\Gamma = \Omega$. A holomorphic bundle automorphism as above, multiplied by a suitable real scale factor, will not only identify the two connections, but also send one fibre metric onto the other.

25. Appendix. Compact product manifolds with (0.1)

The examples described next are well-known, at least for $m = 2$ (cf. [**23**, pp. 343 and 345–350] and [**11**]), and have an immediate generalization (see [**12**]) to the case where $M = N \times S^2$ is replaced by a suitable flat S^2 bundle over N.

Given an integer $m \ge 2$ and a real number $K > 0$, let (M, g) be a product Kähler manifold whose factors are an oriented 2-sphere S^2 with a metric γ of constant Gaussian curvature K, and any compact Kähler-Einstein manifold (N, h) of complex dimension $m-1$ with the negative-definite Ricci tensor $\mathrm{r}^{(h)} = \kappa h$, where $\kappa = (3-2m)K$. Treating our S^2 as the sphere of radius $1/\sqrt{K}$ about 0 in a Euclidean 3-space V, let us choose $\tau : S^2 \to \mathbf{R}$ to be the restriction to S^2 of any non-zero linear homogeneous function $V \to \mathbf{R}$. In other words, τ is an eigenfunction of the spherical Laplacian for the first non-zero eigenvalue $-2K$ (see below). The quadruple (M, g, m, τ), with τ viewed as a function on M constant in the direction of the N factor, then satisfies (0.2). Specifically, the tensor field b on M given by $b = 2(m-1)\nabla d\tau + \tau\mathrm{r}$ equals $(3-2m)K\tau g$, which implies (6.2) with $n = 2m$ (and hence (0.1)), while $\Delta\tau = -2K\tau$ (which gives (0.2)).

In fact, let $\dot{\tau} = d[\tau(x(s))]/ds$ for a fixed C^2 curve $s \mapsto x(s)$ in a Riemannian manifold (M, g) and a C^2 function $\tau : M \to \mathbf{R}$. By (2.2), $\dot{\tau} = g(v, \dot{x})$, where $\dot{x} = dx/ds$ and $v = \nabla\tau$. Applying d/ds again, we obtain $\ddot{\tau} = g(\nabla_{\dot{x}}v, \dot{x}) + g(v, \nabla_{\dot{x}}\dot{x})$. If the curve is a geodesic, that is, $\nabla_{\dot{x}}\dot{x} = 0$, this becomes $\ddot{\tau} = (\nabla d\tau)(\dot{x}, \dot{x})$, by (2.3) with $u = w = \dot{x}$. The last relation, applied to our linear function $\tau : S^2 \to \mathbf{R}$ and all great circles in S^2 with standard sine/cosine parameterizations $s \mapsto x(s)$, yields $\nabla d\tau = -K\tau\gamma$ in (S^2, γ), and hence also in (M, g) (where γ is identified with its pullback to M). Thus, $\Delta\tau = -2K\tau$ in both (S^2, γ) and (M, g).

Finally, to see that $b = (3-2m)K\tau g$, note that the factor distributions in M are r-orthogonal and $\nabla d\tau$-orthogonal, and hence b-orthogonal to each other, while, as $\nabla d\tau = -K\tau\gamma$, the restrictions of $\nabla d\tau$, r and b to the N (or, S^2) factor distribution are 0, $\mathrm{r}^{(h)} = (3-2m)Kh$ and $(3-2m)K\tau h$ (or, respectively, $-K\tau\gamma$, $K\gamma$ and $(3-2m)K\tau\gamma$), which proves our claim.

According to Corollaries 9.3 and 13.2(iii), the quadruples (M, g, m, τ) just described all satisfy condition (a) in Proposition 22.1, and so, by Theorem 24.1, they arise (locally, at points with $d\tau \neq 0$) from the construction of § 23, with Q as in (i) of § 23. More precisely, $Q = g(\nabla\tau, \nabla\tau)$ is given by (19.1) with K as above, $\alpha = 0$, and some $\eta \in (-\infty, 0)$ depending on the choice of the linear function $V \to \mathbf{R}$. In fact, we also have $Q = \gamma(\nabla\tau, \nabla\tau)$ (the γ-gradient in (S^2, γ)) and, since $\nabla d\tau = -K\tau\gamma$ in (S^2, γ), (2.4.ii) and (2.2) applied to (S^2, γ) (rather than (M, g)) give $dQ = -2K\tau d\tau$, so that $Q + K\tau^2$ is a positive constant, as required.

26. Appendix. Bérard Bergery's and Page's examples

Section 8 of Lionel Bérard Bergery's 1982 paper [**2**] describes a family of non-Kähler, Einstein metrics on holomorphic $\mathbf{C}\mathrm{P}^1$ bundle spaces M in all complex dimensions $m \geq 2$, which includes the Page metric (see [**21**]) with $m = 2$. Every metric $\tilde{g}$ in that family is globally conformal to a Kähler metric g (see [**2**]), that is, $\tilde{g} = g/\tau^2$ for some non-constant C^∞ function $\tau : M \to \mathbf{R} \smallsetminus \{0\}$. Our Theorem 24.1 now implies that locally, at points with $d\tau \neq 0$, those g, τ may be obtained via the construction in § 23 (also for $m = 2$, since the additional assumption in (0.2) happens to hold as well).

Instead of using Theorem 24.1, we will now verify the last statement directly, by explicitly showing that Bérard Bergery's construction is a special case of ours.

By saying 'special case' we do not claim that the existence of the *compact* manifolds found by Bérard Bergery easily follows from our results. Our local approach ignores the boundary conditions necessary for compactness which, in Bérard Bergery's construction, amount to a very careful choice of the pair λ, x in the table below; in the exposition that follows, $\lambda = \eta/m$ is arbitrary, and x does not appear at all. (Bérard Bergery's choice of λ, x leads, when N is compact, to a compactified version $(M, \tilde{g})$ of the Einstein manifold $(M', \tilde{g})$ described below.) For compact manifolds that generalize Bérard Bergery's examples, see [**13**].

Since the original paper [**2**] is difficult to obtain, our account of Bérard Bergery's examples follows their presentation in [**3**, Section K of Chapter 9, pp. 273–275]. It is described there how an Einstein metric g is constructed using a real variable t, functions f, h, φ of t, and P of φ, an integer n, a rational constant written as s/q, real constants l, λ, x, a Kähler-Einstein manifold $(B, \check{g})$, and a $\mathrm{U}(1)$ bundle over B with the projection mapping p. The symbols just listed are literally quoted from [**3**]; our outline of the discussion in [**3**] employs different notation, consistent with the rest of the present paper, and related to that in [**3**] as follows:

in [**3**]:	g	t	f	h	φ	P	n	s/q	l	λ	x	B	$\check{g}$	p
below:	$\tilde{g}$	s	Ψ	$(1-\varphi^2)^{1/2}$	φ	P	$2m$	q	ℓ	η/m	(not used)	N	h	π

Specifically, given an integer $m \geq 2$, let $P(\varphi)$ be the unique even polynomial in the real variable φ with $P(0) = 1$ and $d^2P/d\varphi^2 = 2m(1-\varphi^2)^{m-1}$. It follows now that, for E, F as in (21.1) and any $t \in \mathbf{R} \smallsetminus \{0, 1\}$,

$$2(1-2m)E(0)P(\varphi) = (1-\varphi^2)^m[2E(t) - F(t)], \quad \text{where } \varphi = (2-t)/t. \tag{26.1}$$

Namely, $(1-\varphi^2)^m F(t) = -4^m\varphi$, and so (26.1) amounts to $2(1-\varphi^2)^m E(t) +$

$4^m\varphi = 2(1-2m)E(0)P(\varphi)$, which one easily verifies by induction on $m \geq 1$. In fact, for any $m \geq 2$, marking the dependence of E, P on m with a subscript, we have $E_m(t) = 4(1-\varphi^2)^{-1}E_{m-1}(t) + 2(2-3/m)E_{m-1}(0)$ and $mE_m(0)/2 = (2m-3)E_{m-1}(0)$ (which are simply the recursion formulae preceding (21.3)), while $(1-2m)P_m(\varphi) = (1-\varphi^2)^m - 2mP_{m-1}(\varphi)$, which is immediate from the obvious expansion $P_m(\varphi) = 1 + 2m\sum_{k=1}^m(-1)^{k-1}\binom{m-1}{k-1}(2k)^{-1}(2k-1)^{-1}\varphi^{2k}$.

Next, let us choose a complex line bundle $\mathcal{L}$ over a Kähler-Einstein manifold (N,h) of complex dimension $m-1$, with the Ricci tensor $\mathrm{r}^{(h)} = 2mh$, and a C^∞ connection in $\mathcal{L}$, with a horizontal distribution denoted $\mathcal{H}$, which makes some Hermitian fibre metric $\langle\,,\rangle$ parallel and has the curvature form (Remark 3.1) equal to $2mq$ times the Kähler form of (N,h) (see (4.1)), where q is a fixed rational number with $0 < q < 1$.

It follows that $\mathcal{L}$ admits a structure of a holomorphic line bundle for which $\mathcal{H}$ is J-invariant, J being the complex structure tensor on the total space. (See, for instance, [**12**, Remark 2.4].)

Finally, let φ be a function $(0,\ell) \to (-1,1)$ of the real variable $s \in (0,\ell)$, with some $\ell > 0$, for which, writing $\varphi' = d\varphi/ds$ we have, for a suitable constant η,

$$(26.2) \qquad (1-\varphi^2)^{m-1}[\varphi']^2 \,=\, (1-\varphi^2)^m \,+\, (2m-1-\eta/m)\,P(\varphi) \,>\, 0\,.$$

We also fix a positive function r of the variable s such that $\log r$ is an antiderivative of $1/\Psi$ or $-1/\Psi$, that is, $dr/ds = \pm r/\Psi(s)$, where $\Psi = \varphi'/(mq)$. Thus, $s \mapsto r$ is a C^∞ diffeomorphism of $(0,\ell)$ onto a subinterval of $(0,\infty)$, which allows us to treat φ as a function of r. The symbol r also stands for the norm function of $\langle\,,\rangle$, cf. Remark 3.2, so that φ now becomes a function $M' \to \mathbf{R}$ on a suitable open connected subset $M' \subset \mathcal{L} \smallsetminus N$ of the total space of $\mathcal{L}$ (cf. (3.2)).

Any of Bérard Bergery's Einstein metrics $\tilde{g}$ is defined, on this M', so that $\tilde{g} = (1-\varphi^2)\pi^*h$ on $\mathcal{H}$ while $\tilde{g} = [\varphi'/(mqr)]^2\,\mathrm{Re}\langle\,,\rangle$ on the vertical distribution $\mathcal{V}$, and $\tilde{g}(\mathcal{H},\mathcal{V}) = \{0\}$. According to Remark 26.1 below, our description of $\tilde{g}$ on $\mathcal{V}$ is equivalent to that in [**3**], where the restriction $\tilde{\gamma}$ of $\tilde{g}$ to any fibre of $\mathcal{L}$ is defined as in Remark 26.1, with Ψ depending on s as above, that is, $\Psi = \varphi'/(mq)$.

Using a metric $\tilde{g}$ on M' thus obtained, let us now define real constants $a, \varepsilon, c, \kappa$, functions $\tau, Q : M' \to \mathbf{R}$, and a metric g on M', by setting $a = -mq$, $\varepsilon = 1$, $c = 1/2$, $\kappa = 2m$, as well as $Q = \tau^2[\varphi']^2$ and $g = \tau^2\tilde{g}$, where $\tau = ct$ and $t : M' \to \mathbf{R}$ is related to φ by $\varphi = (2-t)/t$. Thus, $1-\varphi^2 = 2\varepsilon(\tau-c)/\tau^2$, and so (26.1), (26.2) give (21.5) for $A = 1$ and some specific B, C, namely

$$(26.3) \qquad \frac{Q}{t-1} \,=\, 1 \,+\, \frac{(2m-1)^{-1}m^{-1}\eta - 1}{2E(0)}\,[2E(t) - F(t)].$$

Bérard Bergery's Einstein metric now has the form $\tilde{g} = g/\tau^2$, with (M',g) and $\tau, Q : M' \to \mathbf{R}$ constructed as in § 23, cf. Proposition 23.3; the function Q of the variable τ used here is of type (iii) in § 23, with $a, \varepsilon, c, \kappa$ defined above.

Remark 26.1. In [**3**] the restriction of the metric $\tilde{g}$ to any fibre of $\mathcal{L}$ appears as a metric on a cylinder, while in this presentation we explicitly define it on an open annulus in the fibre. Equivalence of the two approaches can be seen as follows.

Let S^1 be the unit circle centered at 0 in a one-dimensional complex vector space V with a Hermitian inner product $\langle\,,\rangle$. Any positive C^∞ function Ψ of the real variable s, defined on an interval of the form $(0,\ell)$, gives rise to a metric $\tilde\gamma$ on the cylinder $(0,\ell)\times S^1$ such that for each $s\in(0,\ell)$ (or, $z\in S^1$), $s\mapsto(s,z)$ (or, $z\mapsto(s,z)$) is a $\tilde\gamma$-isometric embedding (or, a closed curve of $\tilde\gamma$-length $2\pi\Psi(s)$, with $\tilde\gamma$-constant speed).

The Riemannian surface $((0,\ell)\times S^1,\tilde\gamma)$ then admits a canonical conformal diffeomorphism onto an annulus $V'\subset V\smallsetminus\{0\}$, centered at 0. In fact, any fixed norm-preserving isomorphic identification $V=\mathbf{C}$ gives rise to the polar coordinates r,θ in V and the corresponding Cartesian coordinates $x=r\cos\theta$, $y=r\sin\theta$. (This θ is not related to θ in the preceding sections.) The standard Euclidean metric $\mathrm{Re}\langle\,,\rangle$ on V then equals $dx^2+dy^2=dr^2+r^2d\theta^2$, while $\tilde\gamma=ds^2+\Psi^2d\theta^2$, where θ, restricted to S^1, serves as a local coordinate for S^1. If we now choose a diffeomorphism $s\mapsto r$ of $(0,\ell)$ onto a subinterval of $(0,\infty)$ such that $dr/ds=\pm r/\Psi(s)$, then $\tilde\gamma=(\Psi/r)^2[dr^2+r^2d\theta^2]$. In other words, the push-forward of $\tilde\gamma$ under the diffeomorphism $(0,\ell)\times S^1\to V'$ given by $(s,z)\mapsto rz$ (with r depending on s as above) equals Ψ^2/r^2 times the Euclidean metric.

27. Appendix. Further integrals of the system (10.1)– (10.4)

The system (10.1)–(10.4) has further interesting integrals: $\kappa=(\lambda Q+\phi Y)/|\phi|$ and $\eta=[\mu+(m-1)\lambda]Z^2+2(2m-1)[\psi+(m-1)\phi]Z-m(2m-1)Q$ (with Z as in Lemma 10.1), in addition to those listed in Lemma 10.1. In fact, wherever $\phi\neq0$, (10.1) gives $d[Y+\lambda Q/\phi]/d\tau=0$ (that is, $d\kappa/d\tau=0$). Similarly, for $m\geq2$, (10.1)–(10.3.i) yield $d\eta/d\tau=0$ wherever $\lambda\neq\mu$, as one sees evaluating $d\eta/d\tau$ from (10.5) and (10.1), and then replacing Z with $2(m-1)(\psi-\phi)/(\lambda-\mu)$.

Geometrically, constancy of κ and η is related to Schur's theorem for the Ricci curvature, since they are proportional to the scalar curvatures of some Einstein metrics: for κ, it is the metric h defined in the proof of Theorem 18.1, under the assumption that τ satisfies (7.1) on a Kähler manifold of complex dimension $m\geq3$ and $\phi\neq0$ everywhere in M' (see Lemma 12.5, with $Q,Y,\mathrm{s},\phi,\psi,\lambda,\mu$ as in Lemma 11.1); while, in the case of η, it is the Einstein metric $\tilde g=g/\tau^2$ for M,g,m,τ satisfying (0.1) with $m\geq3$, or (0.2) with $m=2$, since the scalar curvature $\tilde{\mathrm{s}}$ of $\tilde g$ equals 2η by (6.1) for $n=2m$ combined with Lemma 7.5(i), where $Y=\Delta\tau$, and the equality $Z=\tau$ (cf. (10.3.iii) and Corollary 11.2).

From now on we assume that $Q,Y,\mathrm{s},\phi,\psi,\lambda,\mu$ are C^1 functions on an interval $\mathcal{I}$ of the variable τ, satisfying, except for one subcase in (d), conditions (10.1)–(10.4) with a fixed integer $m\geq2$. Aside from that subcase, we set $\varepsilon=\operatorname{sgn}\phi$ on $\mathcal{I}$ and let c be the constant defined in Lemma 10.1. Then, for κ,η as above,

(a) $\varepsilon\kappa/2$ equals (20.3), as one sees writing $\varepsilon\kappa=Y+\lambda Q/\phi$ and then replacing Y,λ,Q by the expressions in (20.1).
(b) If $c=0$, (20.4.i) and (a) give $\kappa=\varepsilon K$, with K appearing in (20.4).
(c) We have $\eta/m=-\tau^2(\tau-c)\phi''+[(m-1)\tau-2mc]\tau\phi'+2(2m-1)c\phi$, with $\phi'=d\phi/d\tau$, as one sees replacing Z in η by τ (cf. (10.3.iii)), and then using (20.1) to express μ,λ,ψ,Q through ϕ,ϕ',ϕ''.

(d) Under the assumptions (10.1)–(10.4) and $c = 0$ (or, (10.1)–(10.3) and $\phi = 0$), our η coincides with the constant η in (20.4) (or, (19.1)), which is immediate from (c) with $c = 0$ (or, respectively, the relation $Z = \tau$, cf. (10.3.iii), and the expressions for $\mu, \lambda, \psi, \phi, Q$ in § 19).

(e) Let $c \neq 0$ and let A, B, C be as in (21.5). Then, by (a) and the final part of Remark 21.3, $\kappa = \varepsilon m A/c$. On the other hand, $\eta = (2m-1)m[A+BE(0)]$, with E as in (21.1). In fact, by (c), $\eta/m = c\mathcal{P}[\phi]$, where ϕ is treated as a C^2 function of the variable $t = \tau/c$ and, for any such function ϕ, we set $\dot{\phi} = d\phi/dt = c\phi'$ and $\mathcal{P}[\phi] = -t^2(t-1)\ddot{\phi} + [(m-1)t-2m]t\dot{\phi} + 2(2m-1)\phi$. What we need to show is that $c\mathcal{P}[\phi] = (2m-1)[A + BE(0)]$ for every ϕ in the three-dimensional solution space V mentioned in Remark 21.3, with A, B corresponding as in (21.5) to $Q = 2(\tau - c)\phi$, or, equivalently, $\mathcal{P}[\phi] = 2(2m-1)\phi(0)$ for all $\phi \in V$. This amounts to equality between two linear functionals on V, which we can establish using the basis $1, E, F$. The functionals obviously agree on $\phi = 1$. Also, (21.2) with $F(0) = 0$ (see (21.1)) yields $\mathcal{R}[\phi] = 0$ both for $\phi = E$ and $\phi = F$, where $\mathcal{R}[\phi] = t(t-2)\dot{\phi} - \varXi\phi - 2(2m-1)\phi(0)$. Multiplying the equality $\mathcal{R}[\phi] = 0$ by $t(t-1)(t-2)^{-1}$, then applying d/dt, and noting that $t(t-1)(t-2)^{-1}\varXi(t) = mt^2 - 2(m-1)t + 2 + 4/(t-2)$ and $t(t-1)(t-2)^{-1} = t + 1 + 2/(t-2)$, we obtain a formula expressing $-t^2(t-1)\ddot{\phi}$ through ϕ and $\dot{\phi}$. That formula alone now easily gives $\mathcal{P}[\phi] - \mathcal{R}[\phi] - 2(2m-1)\phi(0) = [1 - 2/(t-2)^2]\mathcal{R}[\phi]$, so that $\mathcal{P}[\phi] = 2(2m-1)\phi(0)$ if $\phi = E$ or $\phi = F$, as required.

References

1. V. Apostolov, D. M. J. Calderbank, P. Gauduchon, *The geometry of weakly self-dual Kähler surfaces*, Compositio Math. **135** (2003), 279–322.
2. L. Bérard Bergery, *Sur de nouvelles variétés riemanniennes d'Einstein*, Publ. de l'Institut E. Cartan (Nancy) **4** (1982), 1–60.
3. A. L. Besse, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **10** (Springer, Berlin, 1987).
4. R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. **145** (1969), 1–49.
5. R. Bott and S. S. Chern, *Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections*, Acta Math. **114** (1965), 71–112.
6. E. Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry (ed. S. T. Yau), Annals of Mathematics Studies **102** (Princeton University Press, 1982), 259–290.
7. E. Calabi, *Extremal Kähler metrics, II*, Differential geometry and complex analysis (ed. I. Chavel and H. M. Farkas), a volume dedicated to the memory of Harry Ernest Rauch (Springer, Berlin, 1985), 95–114.
8. H.-D. Cao, *Existence of gradient Kähler-Ricci solitons*, Elliptic and parabolic methods in geometry, Minneapolis, MN, 1994 (ed. B. Chow, R. Gulliver, S Levy and J. Sullivan, A. K. Peters, Wellesley, MA, 1996), 1–16.
9. T. Chave and G. Valent, *Compact extremal versus compact Einstein metrics*, Classical Quantum Gravity **13** (1996), 2097–2108.
10. A. Derdziński, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compositio Math. **49** (1983), 405–433.
11. A. Derdziński, *Hermitian Einstein metrics*, Global Riemannian Geometry, Durham, 1983 (ed. T. J. Willmore and N. Hitchin, Ellis Horwood, Chichester, 1984), 105–114.

12. A. Derdzinski and G. Maschler, *Special Kähler-Ricci potentials on compact Kähler manifolds*, J. reine angew. Math **593** (2006), 73–116.
13. A. Derdzinski and G. Maschler, *A moduli curve for compact conformally-Einstein Kähler manifolds*, Compositio Math. **141** (2005), 1029–1080.
14. P. Griffiths and J. Harris, *Principles of Algebraic Geometry* (Interscience, New York, 1978).
15. S. Gudmundsson, *On the geometry of harmonic morphisms*, Math. Proc. Cambridge Philos. Soc. **108** (1990), 461–466.
16. A. D. Hwang and S. R. Simanca, *Extremal Kähler metrics on Hirzebruch surfaces which are locally conformally equivalent to Einstein metrics*, Math. Ann. **309** (1997), 97–106.
17. A. D. Hwang and M. A. Singer, *A momentum construction for circle invariant Kähler metrics*, Trans. Amer. Math. Soc. **354** (2002), 2285–2325.
18. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry,*, Vol. I (Interscience, New York, 1963).
19. W. Kühnel, *Conformal transformations between Einstein spaces*, Conformal geometry, Bonn, 1985/1986 (ed. R. S. Kulkarni and U. Pinkall), Aspects of Mathematics E12 (Vieweg, Braunschweig, 1988), 105–146.
20. C. LeBrun, *Einstein metrics on complex surfaces,*, Geometry and Physics, Aarhus, 1995 (ed. J. E. Andersen, J. Dupont, H. Pedersen and A. Swann), Lecture Notes in Pure and Applied Mathematics **184** (Dekker, New York, 1997), 167–176.
21. D. Page, *A compact rotating gravitational instanton*, Phys. Lett. B **79** (1978), 235–238.
22. H. Pedersen, C. Tønnesen-Friedman and G. Valent, *Quasi-Einstein Kähler metrics*, Lett. Math. Phys. **50** (1999), 229–241.
23. A. Z. Petrov, *Einstein Spaces* (Pergamon, Oxford, 1969), English translation of *Prostranstva Eynshteyna* (Fizmatlit, Moscow, 1961).
24. Y. Sakane, *Examples of compact Einstein Kähler manifolds with positive Ricci tensor*, Osaka J. Math. **23** (1986), 585–616.
25. S. R. Simanca, *Strongly extremal Kähler metrics*, Ann. Global Anal. Geom. **18** (2000), 29–46.
26. G. Tian and X. Zhu, *Uniqueness of Kähler-Ricci solitons*, Acta Math. **184** (2000), 271–305.
27. C. Tønnesen-Friedman, *Extremal Kähler metrics and Hamiltonian functions II*, Glasgow Math. J. **44** (2002), 241–253.
28. J. Wang and M. Y. Wang, *Einstein metrics on S^2-bundles*, Math. Ann. **310** (1998), 497–526.

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