

Conformally Kähler base metrics for Einstein warped products

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Abstract

A Riemannian metric g with Ricci curvature r is called nontrivial quasi-Einstein, in a sense given by Case, Shu and Wei, if it satisfies $(-a/f)\nabla df + r = \lambda g$, for a smooth nonconstant function f and constants λ and $a > 0$. If a is a positive integer, it was noted by Besse that such a metric appears as the base metric for certain warped Einstein metrics. This equation also appears in the study of smooth metric measure spaces. We provide a local classification and an explicit construction of Kähler metrics conformal to nontrivial quasi-Einstein metrics, subject to the following conditions: local Kähler irreducibility, the conformal factor giving rise to a Killing potential, and the quasi-Einstein function f being a function of the Killing potential. Additionally, the classification holds in real dimension at least six. The metric, along with the Killing potential, form an SKR pair, a notion defined by Derdzinski and Maschler. It implies that the manifold is biholomorphic to an open set in the total space of a CP^1 bundle whose base manifold admits a Kähler-Einstein metric. If the manifold is additionally compact, it is a total space of such a bundle or complex projective space. Additionally, a result of Case, Shu and Wei on the Kähler reducibility of nontrivial Kähler quasi-Einstein metrics is reproduced in dimension at least six in a more explicit form.

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1. Introduction

The long-standing question mentioned in Besse [1], of the existence of compact Einstein nontrivial warped product manifolds, was recently resolved by Lü, Page and Pope [8], who gave the first examples. It is known [7, 1] that for such a product $M \times_f F^a$ to be Einstein, F must itself be an a -dimensional Einstein manifold with Einstein constant equal to $f\Delta f + (a-1)|\nabla f|^2 + \lambda f^2$. Here the Laplacian Δ and the norm $|\cdot|$ are computed with respect to the base metric g , and λ is the Einstein constant of the warped product. A further, crucial requirement is that g will satisfy the equation

$$(-a/f)\nabla df + r = \lambda g \tag{1.1}$$

where r is the Ricci tensor, and ∇d the Hessian. Notice that the parameter a , being a dimension, is constrained in this equation to be a positive integer.

Aside from its usage for Einstein warped products, Equation (1.1) plays a role in the study of smooth metric measure spaces, which are connected with the approach given by Perelman to the Ricci flow. For this one employs the following generalization of the Ricci tensor. On a manifold M of dimension n , the a -Bakry-Emery Ricci tensor of a pair (g, u) , consisting of a Riemannian metric g and a smooth function u , is defined to be the symmetric 2-tensor

$$r_u^a = r + \nabla du - a^{-1} du \otimes du, \quad \text{for a constant } 0 < a < \infty,$$

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where r is the Ricci tensor of g and ∇du is the Hessian of u . If g satisfies the equation $r_u^a = \lambda g$, for $\lambda \in \mathbf{R}$, g was called in [2] a quasi-Einstein metric¹. The limiting value $a = \infty$, where a quasi-Einstein metric becomes a gradient Ricci soliton, will not concern us in this work, and thus, in opposition to the convention of [2], we exclude it from the definition.

An important observation made by Case, Shu and Wei [2] is that the substitution $f = \exp(-u/a)$ (for a finite), converts the equation for a quasi-Einstein metric to the form (1.1) for f . We will call (g, f) a quasi-Einstein pair, and f the quasi-Einstein function. If f is constant, g is Einstein. If (1.1) is satisfied for some nonconstant f , the quasi-Einstein pair is said to be nontrivial.

An attempt to relate quasi-Einstein metrics to Kähler geometry was made in [2], where, however, it was found that there are no nontrivial Kähler quasi-Einstein metrics (with, of course, finite a) on compact manifolds. This result depends on a structure theorem obtained there in the complete simply connected case, but the argument given is essentially local. In an appendix we give, under mild assumptions, another, somewhat more explicit version of this structure theorem.

Our main aim in this work is to address the issue of whether a quasi-Einstein metric can be conformal to a Kähler metric. We answer this question in the affirmative. In fact, using methods akin to those in [9], we give the following classification and existence theorem which relies on the notion of an SKR pair [9], a well understood metric type first studied in [3] (see Definition 3.1).

Theorem A. *On a manifold M of real dimension $n \geq 6$, suppose g is a Kähler, and not a local product of Kähler metrics in any neighborhood of some point. Suppose τ is a nonconstant Killing potential for g , and there exists a nontrivial quasi-Einstein pair of the form $(g/\tau^2, f(\tau))$ on $M \setminus \tau^{-1}(0)$. Then:*

the pair (g, τ) is an SKR pair;

the manifold M is biholomorphic to an open set in the total space of a CP^1 bundle whose base manifold admits a Kähler-Einstein metric;

$f = K\tau^{-1} + L$ for nonzero constants K and L .

Conversely, in all even dimensions $n \geq 4$, there exist SKR pairs (g, τ) and corresponding nonzero constants K and L for which $(g/\tau^2, K\tau^{-1} + L)$ is a nontrivial quasi-Einstein pair.

Note that the proof, relying on the local classification of SKR pairs, yields families, given explicitly, of all possible metrics g (and hence g/τ^2) in Theorem A, up to biholomorphic isometries.

The theory of SKR pairs has been used to obtain other results on Kähler metrics conformal to distinguished metrics. Namely, a similar result holds if g/τ^2 is assumed to be Einstein ([3], see also [10] for examples), without, of course, any suppositions or conclusions on f . Similarly, a completely analogous result holds if g/τ^2 is a gradient Ricci soliton [9]. The present quasi-Einstein case differs from the soliton case in two regards: to carry out the classification in the latter case, it is enough to consider the soliton function $f = \tau^{-1}$ (i.e. $K = 1, L = 0$); more importantly, in the soliton case, g/τ^2 is itself Kähler with respect to an oppositely oriented complex structure. Neither of these properties hold for the quasi-Einstein metrics in Theorem A. On the other hand, in both the soliton and the quasi-Einstein cases, the assumption that τ gives rise to a Killing vector field can be dropped if one requires in advance that f is affine in τ^{-1} .

We stress that while Theorem A is of a local nature, the theory of SKR pairs also yields a classification for compact manifolds. In fact, one has

¹This is in contrast with the use of the term in the physics literature to denote a gradient Ricci soliton. Other usages, especially referring to an equation involving g, r and $du \otimes du$ but not ∇du , also exist [6].

Theorem B. *With all assumptions on M , g , τ and f as in Theorem A, assume also that M is compact. Then either M is biholomorphic to complex projective space, or to the total space of a CP^1 bundle whose base manifold admits a Kähler-Einstein metric.*

This follows directly from global SKR pair theory, specifically [4, Theorem 29.2]. Note further that using local SKR theory, one obtains explicit families for the metric g . To obtain an exact classification of which of these metrics in fact extends to a compact manifold, one must carry out further work analyzing boundary and positivity conditions on the function $Q = g(\nabla\tau, \nabla\tau)$, along the lines of [5]. It is reasonable to expect that the quasi-Einstein metrics appearing in such a global classification will include those of Lü, Page and Pope [8] (because, for example, the compact spaces on which their metrics live are among those named in Theorem B). If this is indeed the case, it will follow that their metrics are conformally Kähler.

The proof of Theorem A proceeds as follows. Consideration of conformal changes yields that the pair (g, τ) satisfies a Ricci-Hessian equation, with coefficients that force it to be a “standard” SKR pair. The construction of such a pair depends on the existence of a nontrivial horizontal eigenfunction for the Hessian of τ . This function is locally a function of τ , and is obtained by solving a pair of linear second order odes. The desired nontrivial solution is obtained when L/K is determined explicitly from constants associated with the SKR pair, in such a way that it is nonzero. Inserting this solution as an ingredient in the canonical construction of an SKR pair yields the desired examples.

Note that Equations (5.5) were obtained using a symbolic computation program.

2. Quasi-Einstein metrics and Ricci-Hessian pairs

On a manifold M of real dimension larger than two, a pair (g, τ) consisting of a Riemannian metric g and a smooth nonconstant function τ is called a *Ricci-Hessian pair* [9] if the equation

$$\alpha \nabla d\tau + r = \gamma g \quad (2.1)$$

holds for the Hessian of τ , the Ricci tensor r of g , and some C^∞ coefficient functions α and γ , possibly defined only on an open set of M .

The pair (g, f) satisfying (1.1), with f nonconstant, is an example of a Ricci-Hessian pair. Another example, and one that will be of main interest in what follows, is provided by the requirement that a conformal change of g yields a particular quasi-Einstein metric. Namely, suppose that g is a Riemannian metric and τ, f are nonconstant functions for which the pair $(\hat{g} = g/\tau^2, f)$ is a quasi-Einstein metric. Thus $(-a/f)\hat{\nabla}df + \hat{r} = \lambda\hat{g}$ holds for a constant λ and a positive constant a , where $\hat{r}, \hat{\nabla}$ are the Ricci form and covariant derivative, respectively, of \hat{g} . Suppose further that $df \wedge d\tau = 0$, so that f is locally a function of τ . Using the formulas $\hat{r} = r + (n-2)\tau^{-1}\nabla d\tau + [\tau^{-1}\Delta\tau - (n-1)\tau^{-2}Q]g$ and $\hat{\nabla}df = \nabla df + \tau^{-1}[d\tau \otimes df + df \otimes d\tau - g(\nabla\tau, \nabla f)g]$ (see [9, (2.1),(2.3)]), a calculation as in [9, (2.9)] gives

$$\begin{aligned} r &+ ((n-2)\tau^{-1} - a f'/f) \nabla d\tau - (a/f) (f'' + 2\tau^{-1}f') d\tau \otimes d\tau \\ &= [\lambda\tau^{-2} - \tau^{-1}\Delta\tau + ((n-1)\tau^{-2} - a f'\tau^{-1}f^{-1}) Q] g. \end{aligned} \quad (2.2)$$

Here $Q = g(\nabla\tau, \nabla\tau)$ and the prime denotes differentiation with respect to τ . If f is affine in τ^{-1} , the coefficient of $d\tau \otimes d\tau$ vanishes, and so we have:

Proposition 2.1. *Suppose τ is a nonconstant smooth function and g is a Riemannian metric that is conformal to a nontrivial quasi-Einstein metric g/τ^2 for which the quasi-Einstein function is affine in the square root of the conformal factor (i.e. in τ^{-1}). Then (g, τ) satisfies a Ricci-Hessian equation. The conclusion also follows if the quasi-Einstein function is merely locally a function of τ , provided that g is Kähler and τ is a Killing potential.*

Here τ is a Killing potential if $J\nabla\tau$ is a Killing vector field, with J the almost complex structure on M . The last part of Proposition 2.1 follows since if f is locally a function of τ , the Kähler and Killing assumptions imply that it is *in fact* affine in τ^{-1} . The proof of this claim is identical to that of [9, Proposition 3.1]). Note that conversely, if g is Kähler and τ is a smooth nonconstant function such that (g, τ) satisfies the Ricci-Hessian equation (2.1), then it follows that τ is a Killing potential on the support of α (see [9, beginning of §3.1]).

We note that if $K\tau^{-1} + L$, with $K \neq 0$, is a quasi-Einstein function, so is $\tau^{-1} + L/K$, for the same metric. This allows one to consider only quasi-Einstein functions of the form $f = \tau^{-1} + k$ for a constant k , as we do from here on. From (2.2), we then have that (g, τ) is a Ricci-Hessian pair with

$$\alpha = [n - 2 + a/(1 + k\tau)]/\tau, \quad \gamma = \lambda\tau^{-2} - \tau^{-1}\Delta\tau + [a/(1 + k\tau) + n - 1]\tau^{-2}Q. \quad (2.3)$$

Of course, this value of α , but with $a = 0$, is the one that occurs when g is conformally Einstein.

3. Relation to SKR pairs

To define the notion of an SKR pair, denote by M_τ the complement, in a manifold M , of the critical set of a smooth function τ . Recall that for a Killing potential τ on a Kähler manifold M , the set M_τ is open and dense in M .

Definition 3.1. [3] *A nonconstant Killing potential τ on a Kähler manifold (M, J, g) is called a special Kähler-Ricci potential if, on the set M_τ , all non-zero tangent vectors orthogonal to $\nabla\tau$ and $J\nabla\tau$ are eigenvectors of both $\nabla d\tau$ and r . The pair (g, τ) is then called an SKR pair.*

The utility of the concept of an SKR pair lies in the fact that it is a classifiable geometric structure [3, 4, 5]. We review part of the theory in Section 4. Here we show:

Proposition 3.2. *On a manifold of real dimension at least six, if (g, τ) is a pair with g Kähler and τ a nonconstant Killing potential, while the associated pair $(g/\tau^2, f(\tau))$ is a nontrivial quasi-Einstein pair, then (g, τ) is an SKR pair. This also holds in real dimension four, provided that $Q = g(\nabla\tau, \nabla\tau)$ and $\Delta\tau$ are locally functions of τ .*

PROOF. By Proposition 2.1 g satisfies a Ricci-Hessian equation

$$\alpha\nabla d\tau + r = \gamma g.$$

The paragraphs past this proposition indicate that f is in fact affine in τ^{-1} , and for the purpose of determining g , it suffices to assume $f = \tau^{-1} + k$, with k a constant.

The coefficients α, γ of the Ricci-Hessian equation are then given by (2.3), from which one sees that α is manifestly a function of τ , i.e. $d\alpha \wedge d\tau = 0$. It then follows that also $d\gamma \wedge d\tau = 0$, if either $n \geq 6$, or else $n = 4$ and both $Q = g(\nabla\tau, \nabla\tau)$ and $\Delta\tau$ are locally functions of τ (see [9, Proposition 3.3 and the paragraph before it]). Finally, it follows from (2.3) that $\alpha d\alpha \neq 0$, except on the sets where $\tau = (2 - n - a)/((n - 2)k)$ or $\tau = \left(- (n - 2 + a) \pm \sqrt{a(n + a - 2)}\right) / ((n - 2)k)$.

These observations imply that the pair (g, τ) is an SKR pair. This follows from [9, Proposition 3.5] away from the above mentioned degeneracy sets, and it follows in the entire τ -noncritical set M_τ by an argument analogous to [9, Corollary 3.7]. \square

4. SKR pair theory and associated differential equations

By [3, Definition 7.2, Remark 7.3], the SKR condition on (g, τ) is equivalent to the existence, on M_τ , of an orthogonal decomposition $TM = \mathcal{V} \oplus \mathcal{H}$, with $\mathcal{V} = \text{span}(\nabla\tau, J\nabla\tau)$, along with four smooth functions ϕ, ψ, β, μ which are pointwise eigenvalues for either $\nabla d\tau$ or r , i.e., they satisfy

$$\begin{aligned} \nabla d\tau|_{\mathcal{H}} &= \phi g|_{\mathcal{H}}, & \nabla d\tau|_{\mathcal{V}} &= \psi g|_{\mathcal{V}}, \\ r|_{\mathcal{H}} &= \beta g|_{\mathcal{H}}, & r|_{\mathcal{V}} &= \mu g|_{\mathcal{V}}. \end{aligned} \quad (4.1)$$

This decomposition is also r - and $\nabla d\tau$ -orthogonal.

Remark 4.1. *By [3, Lemma 12.5], ϕ either vanishes identically on M_τ , or never vanishes there. In the former case only, g is reducible to a local product of Kähler metrics near any point (see [3, Corollary 13.2] and [4, Remark 16.4]). In the latter case, we call g a nontrivial SKR metric.*

Remark 4.2. *For a nontrivial SKR metric, consider $c = \tau - Q/(2\phi)$, with $Q = g(\nabla\tau, \nabla\tau)$, and $\kappa = \text{sgn}(\phi)(\Delta\tau + \lambda Q/\phi)$, regarded as functions $M_\tau \rightarrow \mathbb{R}$. By [3, Lemma 12.5], c is constant on M_τ , and will be called the SKR constant. In any complex dimension $m \geq 2$, we will call a nontrivial SKR metric standard if κ is constant (and also use “standard SKR pair” as a designation for (g, τ)). According to [3, §27, using (10.1) and Lemma 11.1], κ is in fact constant if $m > 2$, so that the designation “standard” involves an extra assumption as compared with “nontrivial” only when $m = 2$.*

The following proposition summarizes a number of results given in [9].

Proposition 4.3. *For any SKR pair (g, τ) , the function ϕ , i.e. the horizontal eigenvalue of $\nabla d\tau$, is locally a C^∞ function of τ on M_τ . Furthermore, the pair satisfies a Ricci-Hessian equation on the open set where $\nabla d\tau$ is not a multiple of g . If the pair is standard, the coefficients α and γ of this Ricci-Hessian equation are locally functions of τ , and the function ϕ satisfies the ordinary differential equation*

$$(\tau - c)^2\phi'' + (\tau - c)[m - (\tau - c)\alpha]\phi' - m\phi = -\text{sgn}(\phi)\kappa/2. \quad (4.2)$$

at points of M_τ for which $\phi'(\tau)$ is nonzero. Also, on the same set,

$$\gamma = \alpha\phi + (\alpha(\tau - c) - (m + 1))\phi' - (\tau - c)\phi'' \quad (4.3)$$

holds.

PROOF. The function ϕ is locally a function of τ by [3, Lemma 11.1a]. The existence of a Ricci-Hessian equation is the second part of [9, Proposition 3.5]. Next, α is locally a function of τ by [9, Remark 3.10], while the same then follows for γ by [9, Proposition 3.3]. Equation (4.2) holds by [9, Proposition 4.1], and the expression (4.3) is obtained in [9, second paragraph of Section 4.2]. \square

We will need the following refinement of this proposition, which extends the domain of definition of the above equations, and was implicitly assumed in [9].

Proposition 4.4. *Any Ricci-Hessian equation satisfied by an SKR pair (g, τ) on some set, coincides with the one of Proposition 4.3 on the intersection of their domains. Furthermore, Equations (4.2) and (4.3) hold on the intersection of the (entire) domain of such a Ricci-Hessian equation with M_τ , if the pair is nontrivial. These two equations are an ode, and, respectively, an expression for γ in terms of functions of τ , if the Ricci-Hessian equation has coefficients that are functions of τ , and the pair is standard.*

PROOF. The coefficients α , γ of the Ricci-Hessian equation in Proposition 4.3 are uniquely determined on the open set of M_τ where $\nabla d\tau$ is not a multiple of g . This follows as r is a unique linear combination of $\nabla d\tau$ and g on this set, as the latter two tensors form at each point of this set a basis for the space of twice covariant tensors for which the nonzero vectors \mathcal{H} and \mathcal{V} are eigenvectors (see [3, Remark 7.4] and [9, Remark 3.6]). Thus any other Ricci-Hessian equation must coincide with the above one, on the intersection of their domains.

Equations (4.2) and (4.3) hold on any subset of M_τ where a Ricci-Hessian equation holds, as their derivation depends only on the relation $\beta - \mu = (\psi - \phi)\alpha$, which holds at all points where the Ricci-Hessian equation holds (see the proofs in [9, Remark 3.10 and Proposition 4.1]). The final statement is immediate. \square

We return now to our standard assumptions, as in, e.g., Proposition 3.2: M is a manifold of real dimension at least six, (g, τ) is a pair with g Kähler and τ a nonconstant Killing potential, while the associated pair $(g/\tau^2, f(\tau))$ is a nontrivial quasi-Einstein pair. (g, τ) forms an SKR pair, by Proposition 3.2. Assume also that g is not reducible as a local product of Kähler metrics in any neighborhood of a given point. In other words, g is nontrivial, and by our assumption on the dimension, (g, τ) is standard (see Remarks 4.1 and 4.2). Proposition 2.1 guarantees the existence of a Ricci-Hessian equation for the pair (g, τ) , with coefficients given by (2.3), defined on the set where $\tau \neq 0$. This set is open and dense in M_τ , as τ is a Killing potential, hence a Morse-Bott function. By Proposition 4.4, Equations (4.2) and (4.3) hold on the intersection of M_τ and $\{\tau \neq 0\}$ (both of which are open dense sets in M).

To consider these equations explicitly, we substitute in Equation (4.2) the expression for α given in (2.3). This yields, after multiplying by $\tau(1+k\tau)$, setting $n = 2m$ and distributing various terms, the equation

$$\begin{aligned} & \tau(\tau - c)^2(1+k\tau)\phi'' + [(2-m)k\tau^3 + (2-m-a+(3m-4)kc)\tau^2] \tau^2 \\ & + ((2a+3m-4)c - 2(m-1)kc^2)\tau - (2m-2+a)c^2] \phi' - [m\tau + mk\tau^2] \phi \\ & = -\text{sgn}(\phi)\kappa\tau(1+k\tau)/2. \end{aligned} \quad (4.4)$$

Similarly, the expression (4.3) for γ (again with the value of α from (2.3)), must equal its expression given in (2.3) on $\{\tau \neq 0\}$. Equating the two expressions, while using $Q = 2(\tau - c)\phi$ (by definition of c) and $\Delta\tau = 2m\phi + 2(\tau - c)\phi'$ ([9, (4.3.c)]) gives

$$\begin{aligned} & \tau^2(\tau - c)(1+k\tau)\phi'' + [(1-m)k\tau^3 + (1-m-a+2mkc)\tau^2] \tau^2 \\ & + c(a+2m)\tau] \phi' + [(a-2c(2m-1)k)\tau - 2c(a+2m-1)] \phi = -\lambda(1+k\tau). \end{aligned} \quad (4.5)$$

We proceed to analyze these equations.

5. Solutions of the equations

To examine the solutions of the system (4.4)–(4.5), we add the product of (4.4) by τ to the product of (4.5) by $-(\tau - c)$. This results in a first order equation, which, after rearrangement of terms yields, together with (4.4), the system

$$\begin{aligned} & \tau(\tau - c)(\tau - 2c)(1+k\tau)\phi' + [-mk\tau^3 - (m+a-2c(2m-1)k)\tau^2] \tau^2 \\ & + (c(3a+4m-2) - 2c^2(2m-1)k)\tau - 2c^2(a+2m-1)] \phi \\ & = (1+k\tau) [-\text{sgn}(\phi)\kappa\tau^2/2 + \lambda(\tau - c)], \end{aligned} \quad (5.1)$$

$$\begin{aligned} & \tau(\tau - c)^2(1+k\tau)\phi'' + [(2-m)k\tau^3 + (2-m-a+(3m-4)kc)\tau^2] \tau^2 \\ & + ((2a+3m-4)c - 2(m-1)kc^2)\tau - (2m-2+a)c^2] \phi' - [m\tau + mk\tau^2] \phi \\ & = -\text{sgn}(\phi)\kappa\tau(1+k\tau)/2. \end{aligned} \quad (5.2)$$

To study solutions of this system, we recall the following ([9, Lemma 4.3])

Lemma 5.1. *Let $\{\phi' + p\phi = q, A\phi'' + B\phi' + C\phi = D\}$ be a system of ordinary differential equations in the variable τ , with coefficients p, q, A, B, C and D that are rational functions. Then, on any nonempty interval admitting a solution ϕ , either*

$$A(p^2 - p') - Bp + C = 0 \quad (5.3)$$

holds identically, or

$$\phi = (D - A(q' - pq) - Bq) / (A(p^2 - p') - Bp + C). \quad (5.4)$$

holds away from the (isolated) singularities of the right hand side.

In applying Lemma 5.1 to the system formed by (5.1) and (5.2), we of course modify (5.1) appropriately, dividing it by the factor $\tau(\tau - c)(\tau - 2c)(1 + k\tau)$. The resulting system has a solution set identical to that of (4.4)–(4.5) (certainly on intervals not containing 0, c , $2c$ and $-1/k$ (if $k \neq 0$), and by a continuity argument, on any interval). Computing (5.3) and the numerator of (5.4) in this case using a symbolic computation program (and also verified by hand for the case $k = 0$), we get

$$\begin{aligned} D - A(q' - pq) - Bq &= 0, \\ A(p^2 - p') - Bp + C &= a(\tau - c)^2(2ck + 1)/((\tau - 2c)(\tau k + 1)). \end{aligned} \quad (5.5)$$

This immediately gives

Proposition 5.2. *Suppose $a(2ck + 1) \neq 0$. Then the system (4.4)–(4.5) has no nonzero solutions on any nonempty open interval.*

PROOF. Assume $a(2ck + 1) \neq 0$. Then the right hand side of the second of Equations (5.5) does not vanish identically on the given interval, and thus so does the left hand side. Hence Lemma 5.1 implies that any solution to the system (5.1)–(5.2) is the ratio of the left-hand sides of the two equations in (5.5), away from the point c . This ratio is the zero function. By continuity, neither the system (5.1)–(5.2), nor the equivalent system formed by (4.4) and (4.5), admits any nonzero solutions on the given interval. \square

6. Solutions for the case $k = -1/(2c)$

If $k = -1/(2c)$, Equations (4.4)–(4.5) take, after multiplying by $2c$ and simplifying, the form,

$$\begin{aligned} &\tau(\tau - c)^2(2c - \tau)\phi'' + [(m - 2)\tau^3 + c(8 - 5m - 2a)\tau^2] \phi' + [m\tau(\tau - 2c)]\phi \\ &+ 2c^2(2a + 4m - 5)\tau - 2c^3(2m - 2 + a)]\phi' + [m\tau(\tau - 2c)]\phi \\ &= \operatorname{sgn}(\phi)\kappa\tau(\tau - 2c)/2, \\ &\tau^2(\tau - c)(2c - \tau)\phi'' + [(m - 1)\tau^3 + 2c(1 - 2m - a)\tau^2] \phi' \\ &+ 2c^2(a + 2m)\tau]\phi' + [2c(a + 2m - 1)(\tau - 2c)]\phi = \lambda(\tau - 2c). \end{aligned} \quad (6.6)$$

We consider these solutions for m a positive integer, positive a , and *nonzero* c (as $2ck + 1 = 0$, neither c nor k are zero).

One notices that these equations admit constant nonzero special solutions: $\kappa/(2m)$ with $\kappa > 0$ for the first equation, and $\lambda/(2c(a + 2m - 1))$ for the second. Such a constant will be a joint solution if these two values are equal (implying that $\lambda/c > 0$).

A basis of joint solutions to the associated homogeneous equations is obtained as follows. Each such solution must also solve the homogeneous first order equation associated with (5.1), which is equivalent to $\phi' + p\phi = 0$, where in the expression for p given before Equation (5.5) one sets $k = -1/(2c)$, i.e.

$$p = \frac{a - 1}{\tau - 2c} + \frac{m}{\tau - c} + \frac{1 - a - 2m}{\tau}.$$

The solutions of this equation are constant multiples of $\exp(-\int p)$, i.e. of

$$(\tau - 2c)^{1-a}(\tau - c)^{-m}\tau^{2m-1+a}.$$

To show that this solution indeed solves the differential equations in (6.6), say the first one, note that substituting $\exp(-\int p)$ into $A\phi'' + B\phi' + C\phi$ yields $\exp(-\int p)(A(p^2 - p') - Bp + C)$, which vanishes since (5.3) vanishes by the second equation in (5.5) (with $2ck + 1 = 0$).

To summarize, the general solution to the system (6.6) has the form

$$\phi = C_1 + C_2(\tau - 2c)^{1-a}(\tau - c)^{-m}\tau^{2m-1+a} \quad (6.7)$$

for an arbitrary constant C_2 , and a constant C_1 equal to both $\kappa/(2m)$ and $\lambda/(2c(a + 2m - 1))$.

7. Local geometry of standard SKR pairs

As in [9], we review here the main case in the geometric classification of SKR metrics. Let $\pi : (\hat{L}, \langle \cdot, \cdot \rangle) \rightarrow (N, h)$ be a Hermitian holomorphic line bundle over a Kähler-Einstein manifold of complex dimension $m - 1$. Assume that the curvature of $\langle \cdot, \cdot \rangle$ is a multiple of the Kähler form of h . Note that, if N is compact and h is not Ricci flat, this implies that \hat{L} is smoothly isomorphic to a rational power of the anti-canonical bundle of N .

Consider, on $\hat{L} \setminus N$ (the total space of \hat{L} excluding the zero section), the metric g given by

$$g|_{\mathcal{H}} = 2|\tau - c|\pi^*h, \quad g|_{\mathcal{V}} = \frac{Q(\tau)}{(br)^2} \operatorname{Re} \langle \cdot, \cdot \rangle, \quad (7.1)$$

where

- \mathcal{V}, \mathcal{H} are the vertical/horizontal distributions of \hat{L} , respectively, the latter determined via the Chern connection of $\langle \cdot, \cdot \rangle$,
- c and $b \neq 0$ are constants,
- r is the norm induced by $\langle \cdot, \cdot \rangle$,
- $\tau = \tau(r)$ is a function on $\hat{L} \setminus N$, obtained by composing with r another function, denoted via abuse of notation by τ , and obtained as follows: one fixes an open interval I and a positive C^∞ function $Q(\tau)$ on I , solves the differential equation $(b/Q) d\tau = d(\log r)$ to obtain a diffeomorphism $r(\tau) : I \rightarrow (0, \infty)$, and defines $\tau(r)$ as the inverse of this diffeomorphism.

The pair (g, τ) , with $\tau = \tau(r)$, is a nontrivial SKR pair (see [3, §8 and §16], as well as [4, §4]), $|\nabla\tau|_g^2 = Q(\tau(r))$ holds, and the connection on \hat{L} is not flat. The constant κ of Remark 4.2 is the Einstein constant of h , so that as g is nontrivial, it follows from the stipulation of a Kähler-Einstein base that it is in fact standard (this is opposed to the case of an arbitrary SKR pair, in which h need not be Einstein if $m = 2$). For any g standard, or merely nontrivial, the SKR constant c (see again Remark 4.2) coincides with c of (7.1).

Conversely, for any standard SKR pair (g, τ) on a complex manifold (M, J) , any point in the τ -noncritical set M_τ has a neighborhood which is biholomorphically isometric to an open set in some triple $(\hat{L} \setminus N, g, \tau(r))$ as above. This claim is a special case of [3, Theorem 18.1]). The biholomorphic isometry identifies $\operatorname{span}(\nabla\tau, J\nabla\tau)$ and its orthogonal complement, with \mathcal{V} and, respectively, \mathcal{H} . Moreover, whenever one can extend some $(g, \tau(r))$ to all of \hat{L} , such a biholomorphic isometry can also be defined on neighborhoods of points in $M \setminus M_\tau$ [4, Remark 16.4].

8. Conclusion

The proof of Theorem A can now be concluded as follows. Given a manifold M of dimension at least six, with g Kähler and $(g/\tau^2, f(\tau))$ a nontrivial quasi-Einstein pair, it follows Proposition 3.2 that (g, τ) is an SKR pair. Since g is Kähler-irreducible, by Remark 4.1 this SKR pair is nontrivial, and with the assumption on the dimension, it is standard, according to Remark 4.2. Thus, according to the classification given in Section 7, M is biholomorphic to an open set in the total space of a CP^1 bundle whose base manifold admits a Kähler-Einstein metric (and in fact much more is known about this bundle).

Following the paragraph past Proposition 2.1, the fact that g is Kähler, τ is Killing and f is locally a function of τ implies that f is affine in τ^{-1} , i.e. $f = K\tau^{-1} + L$. The constant K is nonzero since $(g/\tau^2, f)$ is a *nontrivial* quasi-Einstein pair. Now replace f by $\tau^{-1} + k := \tau^{-1} + L/K$, which is another quasi-Einstein function for the same metric g/τ^2 .

According to [3, Lemma 11.1a], the horizontal eigenfunction ϕ of $\nabla d\tau$ is locally a function of τ , and by Proposition 4.4 and the last paragraphs of Section 4, it satisfies the system (4.4)–(4.5) at points of the τ -noncritical set M_τ for which τ is nonzero. Using again Remark 4.1, ϕ is nowhere vanishing on M_τ . This together with Proposition 5.2 implies, since $a \neq 0$, that

$2ck + 1 = 0$, for the constant c associated to the SKR pair (g, τ) as in Remark 4.2. This implies $k \neq 0$, and hence L is nonzero.

Finally, for any $m \geq 2$, fix a choice of data $\pi : (\hat{L}, \langle \cdot, \cdot \rangle) \rightarrow (N, h)$, c nonzero and b satisfying all the criteria required in defining a (standard) SKR pair, and also fix a constant $a > 0$. Using these values of m , c , and a , form the system of equations (6.6), with λ chosen so that the system admits constant solutions. Choose now some solution for ϕ of the form (6.7). For this solution, define $Q(\tau) = 2(\tau - c)\phi(\tau)$, and choose an interval I where Q is positive. On this interval, solve $(b/Q) d\tau = d(\log r)$ to obtain $\tau(r)$ as the inverse of $r(\tau)$. Using $Q(\tau(r))$, $\tau(r)$ and the other data, obtain a standard SKR pair (g, τ) , with τ given by $\tau(r)$, and the metric g given by (7.1). Set $k = -1/(2c)$ and form α according to (2.3), and γ by expression (4.3) (with the value of α just described, and the chosen solution ϕ). As $\phi(\tau)$ is a solution of (6.6), it is also a solution for the equivalent system (4.4)–(4.5) with the choice of k above. By the construction of Equation (4.5), the expression just obtained for γ equals that in (2.3) (with Q and $\Delta\tau$ given using their expressions just before (4.5), which are valid for any nontrivial SKR pair). As (2.3) holds for the SKR pair just constructed, so does (2.2) for $f = (\tau(r))^{-1} - 1/(2c)$. This means exactly that for the SKR pair $(g, \tau(r))$, the pair $(g/(\tau(r))^2, (\tau(r))^{-1} - 1/(2c))$ is a nontrivial quasi-Einstein pair. This concludes the proof of Theorem A.

Appendix A. Local obstructions to the existence of Kähler quasi-Einstein metrics

In [2] it was shown that there exist no nontrivial Kähler quasi-Einstein metrics on compact manifolds. This was based on a structure theorem given for complete nontrivial Kähler quasi-Einstein metrics on simply connected manifolds. While the latter theorem involves global assumptions, the proof is mainly local. Here, using the same methods employed in the proof of Theorem A, we describe, under an extra hypothesis on the dimension, an alternative approach to this result. As it is based on the theory of SKR pairs, it has the merit of leading to explicit expressions for the metric.

Theorem 1. *On a manifold M of complex dimension $m > 2$, suppose g is a Kähler metric and f a nonconstant function such that (g, f) is a (nontrivial) quasi-Einstein pair. Then g is reducible as a local product of Kähler metrics, one of whose components is Kähler-Einstein, and the other situated on a two dimensional manifold, and given explicitly.*

PROOF. The pair (g, f) satisfies the Ricci-Hessian Equation (2.1), with $\alpha = -a/f$ and γ equal to the constant λ . Hence $d\alpha \wedge df = 0$, $d\gamma \wedge df = 0$ and $\alpha d\alpha \neq 0$ on points of the f -noncritical set M_f where $f \neq 0$. Thus by [9, Proposition 3.5], (g, f) is an SKR pair (even for $m = 2$), and so M is biholomorphic to an open set in the total space of a CP^1 bundle whose base manifold admits a Kähler-Einstein metric. We show that the SKR pair cannot be nontrivial. If it is nontrivial, as $m \geq 3$ it is standard, and so by Proposition 4.4, the ode (4.2) (with the above expression substituted for α), holds for the (nonzero) horizontal eigenfunction ϕ of ∇df , at points of M_f for which f is nonzero. Similarly, Equation (4.3) also holds there. The resulting system is

$$\begin{aligned} f(f - c)^2 \phi'' + (f - c)[mf + (f - c)a] \phi' - mf\phi &= -\text{sgn}(\phi)\kappa f/2. \\ -f(f - c)\phi'' + [-a(f - c) - (m + 1)f] \phi' - a\phi &= \lambda f. \end{aligned} \quad (\text{A.1})$$

Adding the first equation to $f - c$ times the second gives the first order equation

$$-f(f - c)\phi' - (a(f - c) + mf)\phi = f(\lambda(f - c) - \text{sgn}(\phi)\kappa/2). \quad (\text{A.2})$$

Calculating the left hand side of (5.5) for the pair consisting of the second equation in (A.1) and Equation (A.2) yields $A(p^2 - p') - Bp + C = -a(f - c)/f$, $D - A(q' - pq) - Bq = 0$. Thus, as $a > 0$, there are no nowhere vanishing solutions ϕ , by Lemma 5.1, and hence the SKR pair cannot be nontrivial. If the SKR pair is trivial, it follows from the classification in [3, Theorem 18.1] of

such pairs, that g is reducible in the manner described in the theorem. Furthermore, the metric is given explicitly by a formula analogous to (7.1), namely $g|_{\mathcal{H}} = \pi^*h$, $g|_{\mathcal{V}} = \frac{Q(\tau)}{(br)^2} \operatorname{Re} \langle \cdot, \cdot \rangle$, with h the Kähler-Einstein metric and Q having an explicit expression in τ (see [3, (19.1)]). \square

Note that the only possibility not explored above is that of a nontrivial SKR pair which is not standard, a possibility which can occur only if $m = 2$. One can investigate this further within SKR theory; however, this possibility is ruled out by the result in [2].

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