Clark University

Math 105 History of Mathematics

First Test Answers

Feb 2014

Scale. 86–100 A, 70–85 B, 55–69 C. Median 86.

Problem 1. Essay. [30] Select *one* of the three topics A, B, and C.

Topic A. Explain the logical structure of the *Elements* (axioms, propositions, proofs). How does this differ from earlier mathematics of Egypt and Babylonia? How can such a logical structure affect the mathematical advances of a civilization?

The *Elements* begins with definitions and axioms. Some of these just describe the terms to be used, others are more substantive and state specific assumptions about properties of the mathematical objects under study. Definitions are given for new concepts stated in terms of the old concepts as the new concepts are needed. Propositions are stated one at a time only using those terms already introduced, and each proposition is proved rigorously. The proof begins with a detailed statement of what is given and what is to be proved. Each statement in the proof can be justified by previous statements, axioms, previously proved propositions, or as an assumption in the beginning of a proof by contradiction. The last statement in a proof is that which was to be proven.

The preHellenic mathematics of Egypt and Babylonia consisted of tables and of solutions to problems of various types. The solutions were meant to describe the methods to solve the problems, what we now call algorithms. No indication was given why the methods should work.

A strict logical structure is primarily needed to convince the audience of the validity of the theory, but it has other purposes. It is used to find flaws in arguments, and even in previously accepted statements. It can be used to find hidden assumptions. More importantly, adhering to a strict logical structure suggests new concepts and new results. Egyptian mathematics reached its high point early in the history of Egypt, about 2000 BCE, and did not progress past that. Babylonian mathematics reached its high point about the same time, 1800 BCE, the Old Babylonian Empire, and also failed to progress after that. The mathematics of both cultures was directed to solving problems. Greek mathematics, on the other hand, progressed as logic developed to the time of Euclid, and continued to progress for several centuries as we will see.

Topic B. Compare and contrast the arithmetic of the Babylonians with that of the Egyptians. Be sure to mention their numerals, algorithms for the arithmetic operations, and fractions. Illustrate with examples. Don't go into their algebra or geometry for this essay.

Here's an list of topics that could go into an essay, but they're not all needed in an essay: Numerals: Egypt — base 10, hieroglyphics repeated symbols for 1, 10, 100, etc, hieratic abbreviations; Babylonia – base 60, positional numeration but used only two marks. Addition and subtraction similar to ours in both cultures. Multiplication: Egypt — repeated duplication two columns (give example); Babylonia — long multiplication similar to ours. Division: Egypt — virtually the same as multiplication, but order of selecting rows different; Babylonia — use a table of reciprocals then multiply. Square roots: Egypt — unknown; Babylonia — tables and fast algorithm. Fractions: Egypt — unit fractions, doubling table for fractions, extend multiplication and division algorithms to fractions; Babylonia — positional numeration like our decimal fractions, but rarely indicate where the decimal point is, same arithmetic algorithms as for whole numbers, but complicated by mentally keeping track of decimal point. One possible essay summary: Babylonians had much more efficient notation and algorithms, could deal with fractions much better.

Topic C. Aristotle presented four of Zeno's paradoxes: the Dichotomy, the Achilles, the Arrow, and the Stadium. Select one, but only one, of them and write about it. State the paradox as clearly and completely as you can. Explain why it was considered important. Refute the paradox, either as Aristotle did, or as you would from a modern point of view.

One of the arguments, the Stadium, depends on conceiving a line as being made out of points in a row, one next to another. Even in Euclidean geometry, there are points on a line, but they do not have that arrangement. The Arrow also depends on time being composed of instants, but not explicitly arranged in a row, one next to another. It does depend, however, in assuming that motion can be determined at an instant without looking at positions at other instants. Aristotle refutes these paradoxes by denying lines are composed of points and time of instants, and by allowing motion only over an interval of time. Aristotle's solution to these two paradoxes differs from that of modern mathematics.

The Dichotomy and the Achilles assert an infinite sequence of occurrences in a finite amount of time. The arguments leading to these occurrences are different in the two paradoxes, but Zeno apparently denied an infinite number of instants in a finite interval of time. Aristotle and the modern point of view agree here. There is no paradox in assuming that there are an infinite number of points on a finite line, or that there are an infinite number of points in time in a finite interval of time.

See the text for more details.

Problem 2. [10] Find the greatest common divisor of the two numbers 11484 and 7902 by using the Euclidean algorithm. (Computations are sufficient, but show your work. An explanation is not necessary.)

For the Euclidean algorithm repeatedly subtract the smaller number from the larger to get smaller and smaller numbers until the smaller divides the larger. Subtract 7902 from 11484 to get 3582. Then work with 7902 and 3583. Now you can subtract 3582 from 7902 a couple times to get a remainder of 738. (That's the same as dividing 7902 by 3582 and keeping the remainder.) Then subtract 738 repeatedly from 3582 to get a remainder of 630. Then $738 - 630 = 108$, and 108 repeatedly subtracted from 630 gives 90, and $108 - 90 = 18$. Finally, 18 divides 90, so 18 is the greatest common divisor.

Problem 3. [10] Multiply the Egypian fractions $7\ \overline{2}\ \overline{4}\ \overline{8}$ by $12\ \overline{3}$ using the Egyptian multiplication technique. Note that it is necessary to multiply each term of the multiplicand by $\overline{3}$ separately.

1	$7\ \overline{2}\ \overline{4}\ \overline{8}$
2	$15\ \overline{2}\ \overline{4}$
′4	$31\ \overline{2}$
′8	63
′$\overline{3}$	$2\ \overline{3}\ \overline{6}\ \overline{12}\ \overline{24}$

Summing 31, 63, and 2 gives 96. Summing $\overline{2}, \overline{3}, \overline{6}$ gives 1, and you can simplify $\overline{12}, \overline{24}$ as $\overline{8}$. So the product, simplified, is $97\ \overline{8}$.

Problem 4. [15] Explain the origins of the sexagesimal system (base 60) used in Mesopotamia. (A one-paragraph answer is sufficient.)

A base 60 system existed in Mesopotamia and the surrounding region when writing developed in 3100 B.C.E. Marks were made with a stylus on cuneiform. It derived from the token system for measuring grain which was a base 60 system that was already thousands of years old by that time.

(The suggestion that 60 has a large number of divisors could possibly be the reason grain was measured using a base 60 system, but that argument does not distinguish 60 from many other possible bases such as 6, 12, 18, 20, 30, 210, etc. There is no evidence that 60 was used because of 12 knuckles on one hand and 5 fingers on the other; it was a base 60 compounded of 6 and 10, not 5 and 12.)

Problem 5. [15] On areas of circles.

a. How did the Egyptians and Babylonians compute the areas of circles?

One Egyptian method that was quite accurate was to take the square on $\frac{8}{9}$ of the diameter. For a circle of diameter 9, that's 64 square cubits.

The Babylonians often used a method that we can summarize as approximating π by 3, equivalently, the area of a circle is approximately $\frac{3}{4}$ the area of the square on the diameter. For a circle of diameter 9, that's $\frac{3}{4}$ of 81, or 60 plus $\frac{3}{4}$, or, as the Egyptians would have expressed it, 60 plus $\frac{1}{2}$ plus $\frac{1}{4}$. (In at least one case, however, the Babylonians recognized a more accurate estimate, namely, 3 1/8 times the area of a square on the radius.)

b. What is the problem of quadrature of circle that vexed the Greeks so much?

That problem is to construct a square equal to (i.e., with the same area as) a given circle. That is, to find the area of a circle. It vexed them because they wanted the construction to use the simplest possible tools; straightedge and compass desirable (but impossible); higher curves are necessary.

(That π is irrational is only part of it. $\sqrt{2}$ is irrational, but the Greek geometers had no difficulty in constructing that.)

Problem 6. [20; 4 points each part] True/false.

a. Euclid's parallel postulate (Postulate 5 in Book I of the *Elements*) stated that lines in the same direction are parallel. /it False. It says if the interior angle sum of a transversal is less than two right angles, then the two lines meet.

b. The ancient Babylonians knew the Pythagorean theorem (the square on the hypenenuse of a right triangle is equal to the sum of the squares on the other two sides) over a thousand years before Pythagoras. /it True.

c. Each of the propositions in Euclid's *Elements* includes a proof. /it True.

d. A triangular number is the perimeter of an equilateral triangle, for example, 15 is a triangular number since an equilateral triangle of side length 5 has perimeter 15. /it False. Triangular numbers have nothing to do with perimeters.

e. Whereas Egyptians used common fractions like $\frac{2}{5}$, Babylonians prefered unit fractions like one-third plus one-fifteenth. /it False.