Rational Approximation

Math 105 History of Mathematics

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Early in the study of the history of mathematics, we come across various rational approximations to irrational numbers. What I mean by that is that when what we know to be an irrational number was used in a computation, a rational number was used instead. A well known example of this is Archimedes' approximation $\frac{22}{7}$ of the irrational number π. Of course, $\frac{22}{7}$ is not exactly equal to π, but it's pretty close.

The question is, how did Archimedes ever come up with this approximation? The answer is known how Archimedes and the Greeks before him found these approximations, but you may not have seen it before. It's sometimes called *antenaresis* or the method of continued fractions, and it's related to Euclid's algorithm for finding the greatest common divisor.

Long before Archimedes, others used rational approximations in Egypt, Babylonia, India, and China. They may have used the same method, but in those cases, there is no record of the method. There are other methods that have been used in specific instances, but when you see a denominator, like 7, that has nothing to do with the numerical base—usually 10, sometimes 60—that's a good indication that antenaresis was used.

What is this method? Let's take an example to see how it goes. Suppose we're trying to "square the circle," that is, find a square whose area equals the area of a given circle. The ratio of the side of the resulting square to the diameter of the original circle will be an irrational number, so an approximation will have to be used in computations.

We've seen that the Egyptians used the square on $\frac{8}{9}$ of the diameter, and a different approximation was used in India, so this particular problem was an important one for the ancients.

First of all, some geometric method is required to give some kind of numerial approximation, then the antenaresis method will give a nice rational approximation that can be used in practice. Let's suppose that somehow we've discovered a good decimal approximation. Since we know the area of a circle to be πr^2, therefore the side s of an equal square will be

$$s = \sqrt{\pi r^2} = r\sqrt{\pi} = \frac{d\sqrt{\pi}}{2},$$

so the ratio of the side s of the square to the diameter d of the circle will be $\frac{1}{2}\sqrt{\pi}$. That means, a good decimal approximation will be 0.886227.

Now, the question is, what's a good rational approximation to 0.886227?

Well, the first approximation, which isn't very good, is the nearest integer, namely 1. That's not a very good approximation since it approximates the circle by the circumscribed square, which is much too big. We want a better approximation.

How far off is our first approximation? Well, $1 - 0.886227$ equals 0.113773. Can we approximate 0.113773 with a nice rational number? It's about $\frac{1}{10}$, or is $\frac{1}{9}$ a better approximation? Here's the trick. Reciprocate that number. We get $1/0.113773 = 8.789432$. Since 8.7 is closest to the integer 9, that means 0.113773 can be pretty well approximated by $\frac{1}{9}$. Now we have our second approximation: 0.886227, which is $1 - 0.886227$ can be approximated by $1 - \frac{1}{9}$, which is $\frac{8}{9}$. That's the ancient Egyptian approximation.

Does this mean that the Egyptians used antenaresis? Not necessarily, but at least it's a possibility.

How do we get the next approximation? Let's look at the error we got with the last approximation. We approximated 8.789432 by 9, which was too large by 0.210568. Use the reciprocation trick again. We get $1/0.210568 = 4.790597$, which is pretty close to 5. So a better approximation than for 8.789432 is $9 - \frac{1}{5}$. That means a better approximation than $1 - \frac{1}{9}$ for 0.886227 is

$$1 - \frac{1}{9 - \frac{1}{5}} = 1 - \frac{5}{44} = \frac{39}{44}.$$

There is no known ancient instance of using $\frac{39}{44}$ for approximating this ratio.

This method works for better and better rational approximations, so long as you have good data going into it. If you don't have a very good approximation to begin with, then only the first couple of stages of antenaresis are useful. The number of stages you use should depend on the degree of accuracy of your approximation.

If you're interested in approximating this particular ratio, here's a comparison table for the first few stages.

$$\begin{aligned} 1 &= 1.0 \\ \frac{8}{9} &= 0.888\ldots \\ \frac{39}{44} &= 0.88636363\ldots \\ \frac{148}{167} &= 0.8862275449 \\ \frac{1}{2}\sqrt{\pi} &= 0.8862269254 \end{aligned}$$