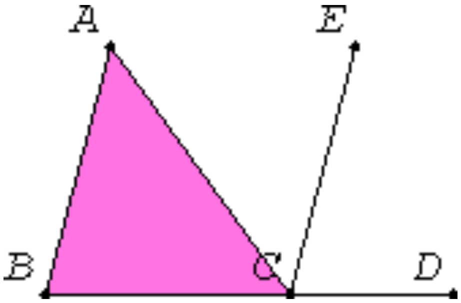
Assignment 4 answers
Math 105 History of Mathematics
Prof. D. Joyce, Spring 2015

On the *Elements*. Page 90, exercises 6, 7, 14, 17, 19.

Exercise 6. Prove Proposition I.32, that the three interior angles of any triangle are equal to two right angles. Show that the proof depends on I.29 and therefore on postulate 5.

Here's Euclid's proof. Yours may be different, but any proof must rely somehow on the parallel postulate because it's known that in hyperbolic geometry Prop. I.32 is false.

First note that proposition I.29 says that when a line crosses two other lines making the alternate interior angles equal, then corresponding angles areequal and the interior angles on the same side of the line are supplementary (add up to two right angles).



Let ABC be the triangle. Extend BC, and draw CE parallel to AB. That's the construction in the previous proposition I.31. Then by I.29, angle B equals angle ECD, and angle A equals angle ACE. Thus the interior angles of the triangle sum to angle CBA plus angle ECD plus angle ACE. But that's a straight angle, the sum of two right angles. Q.E.D.

Comment. In hyperbolic geometry, one of the noneuclidean geometries, the parallel postulate is false. Given a line and a point not on that line, unlike Euclidean geometry in which there is exactly one line through the given point that doesn't meet the given line, in hyperbolic geometry there are infinitely many lines through the point that don't meet the given line.

Also, whereas in Euclidean geometry the angle sum of a triangle equals exactly two right triangles, in hyperbolic geometry the angle sum is always less than two right triangles. In fact, the amount that the angle sum is less than two right triangles, called the *deficiency of triangle* is proportional to the area of the triangle. So for really big triangles, the angle sum is nearly 0.

Exercise 7. Solve the (modified) problem of Proposition I.44, to apply a given straight line AB a rectangle equal to a given rectangle c. Use the supplied figure.

The given rectangle with area c is the rectangle $BEFG$ and the given straight line is AB where ABE is a straight line.

We're to find a rectangle $ABML$ one side of which is AB and the area is also equal to c.

Here's Euclid's proof in Book I. It just uses elementary concepts from that book. After the given figure has been constructed, you can see three pairs of congruent triangles, namely, large triangles HFD and HLD, medium triangles HAB and HGD, and small triangles BED and BMD. But each of the rectangles $BEFG$ and $ABML$ are equal to one of the large triangles minus the sum of one of each of the small and medium triangles. Therefore the rectangles are equal. Q.E.D.

There's a shorter proof involving similar triangles that Euclid didn't give because similar triangles weren't introduced until Book VI, the book after Book V which covered the theory of proportions.

For this proof, note that that the triangles HAB and BMD are similar, so we get the proportion

$$HA : AB = BM : MD.$$

Cross multiplying, we get $HA \cdot MD = AB \cdot BM$. But $HA \cdot MD$, which is equal to $GB \cdot BE$, is the area of one of the rectangles, while $AB \cdot BM$ is the area of the other rectangles. Therefore the rectangles are equal. Q.E.D.

Exercise 14. Prove Proposition III.31, that the angle in a semicircle is a right triangle.

By the way, this is sometimes called Thales' theorem.

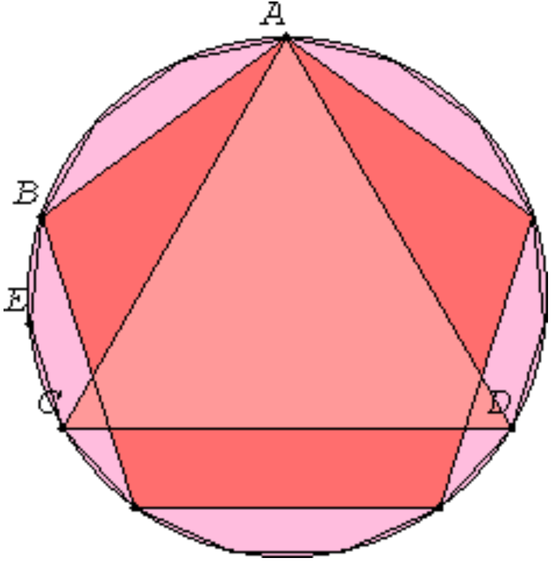
Here's how Euclid did it. Let the triangle be ABC with BC the diameter of the semicircle. Let D be the midpoint of BC which is the center of the circle. Draw AD to get two triangles ADC and ADB.

These two triangles are isosceles triangles since in each case two of the sides are radii. Therefore angle BAC, which equals the sum of the angles BAD and CAD, also equals the sum of the angles at B and C. That is, in the original triangle ABC we have the angle at A is the sum of the other two angles. But the sum of all three angles is 2 right angles (i.e. 180°), and A is half of that, so it's a right angle. Q.E.D.

There's a shorter proof that Euclid didn't give because he didn't accept straight angles as being angles. The straight angle BDC at the center of the circle cuts off the other half of the semicircle, so the angle BAC at the circumference is half of that, and half of a straight angle is a right angle. Q.E.D.

Exercise 17. Given that a pentagon and an equilateral triangle can be inscribed in a circle, show how to inscribe a regular 15-gon in a circle.

See proposition IV.16.



Here's what Euclid did.Construct the equilateral triangle and the regular pentagon in the circle. Let AC be one side of the triangle and AB one side of the pentagon. Bisect the arc BC at E. Then BE and EC are two adjacent sides for the 15-gon. Just repeatedly cut off arcs of that size from the circle to get the rest of the 13 sides.

A more symmetric way is to place the pentagon in the circle, then at each of the 5 vertices of it, place a triangle with one of its vertices at that vertex. The 15 points on the circle are the 15 points of the regular 15-gon.

Exercise 19. Use the Euclidean algorithm to find the greatest common divisor of 963 and 657; of 2689 and 4001.

See proposition VII.2.

There are two versions of this algorithm. The first only uses subtraction. For it, repeatedly subtract the small number from the larger if you only want to use the operation of subtraction. Stop when the two numbers you get are the same.

For 963 and 657, subtract 657 from 964 to get 306.

For 657 and 306, subtract 306 twice from 657 to get 45.

For 306 and 45, subtract 45 from 306 six times to get 36.

For 45 and 36, subtract to get 9

For 36 and 9, subtract three times to get 9. Since both numbers are 9, we've shown that 9 is the GCD of 963 and 657.

The other version of the algorithm involves division. Repeatedly divide the smaller number into the larger and replace the larger by the remainder. Stop when there is no remainder.

For 4001 and 2689, divide 2689 into 4001 giving quotient 1 and remainder 1312.

For 2689 and 1312, divide 1312 into 2689 giving quotient 2 and remainder 65.

For 1312 and 65, divide 65 into 1312 giving quotient 20 and remainder 12.

For 65 and 12, divide 12 into 65 giving quotient 5 and remainder 2.

For 5 and 2, divide 2 into 5 giving quotient 2 and remainder 1.

Stop since 1 divides 2 with no remainder. Therefore, 1 is the GCD of 4001 and 2689. That means they're relatively prime.

Math 105 Home Page at
`http://math.clarku.edu/~djoyce/ma105/`