Math 114 Discrete Math

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Scale: 85–100 A, 70–84 B, 55–69 C. Median 77.

Problem 1. On set operations. [12; 4 points each part] Let $A = \{0, 1, 2, 3\}$, $B = \{0, 1, 4, 5\}$, and $C = \{0, 2, 4, 6\}$. Find

a. $A \cup (B \cap C)$.

The intersection, $B \cap C$, is the set $\{0, 4\}$, so taking the union of A with that gives

$$A \cup (B \cap C) = \{0, 1, 2, 3, 4\}.$$

b. $A \oplus (B - C)$.

$B - C = \{1, 5\}$, so when take the symmetric difference with A and that, we'll get

$$A \oplus (B - C) = \{0, 2, 3, 5\}.$$

c. $(A \cup B \cup C) - (A \cap B \cap C)$.
This is $\{0, 1, 2, 3, 4, 5, 6\} - \{0\} = \{1, 2, 3, 4, 5, 6\}$.

Problem 2. On algorithms and their complexity. [20; 10 points each part] Choose either the linear search algorithm or the binary search algorithm when you answer this question. Your choice.

To standardize notation, suppose you are looking for the value V among $A_1, A_2, \ldots, A_n$. For the binary search, assume that the array is sorted in increasing order.

a. Describe the algorithm either in English or in pseudocode. Be sure to include the possibility that V is not one of the values $A_1, A_2, \ldots, A_n$.

Linear search: Look at each A_i in sequence starting with the first. Stop when you find one equal to V, or, if you don't find one, stop after looking at the last and conclude V is not there.

Binary search: Repeatedly cut the list in (approximately) half until either you find the value or determine it's not there as follows. When you're looking at the list from A_i through A_j, examine the middle element A_m where $m = \lfloor (i+j)/2 \rfloor$. If $A_m = V$, then stop since you found it. If $A_m > V$ then cut the list in half so that you next look at the list from A_i through A_{m-1}, but if $A_m < V$, then cut the list in half so you look only at A_{m+1} through A_j. But if after you repeated half the length of the list no element remains, then conclude that V is not there.

b. Describe the worst case time complexity of the linear search algorithm, that is, state the order of the time it takes the algorithm to execute ($\mathcal{O}(n), \mathcal{O}(\log n), \mathcal{O}(n \log n)$ or whatever). Then in a sentence or two explain why that's correct.

Linear search: In the worst case, you have to go through the entire list and determine V is not there. Thus, $\mathcal{O}(n)$.

Binary search: In the worst case, you have to repeatedly cut the list in half and find it's not there. If n is about 2^k, that means about k times you have to cut it in half. Since $k = \log_2 n$, that gives $\mathcal{O}(\log n)$.

Problem 3. On the growth of functions. [20] Use the definition for big-$\mathcal{O}$ notation to show that $f(x) = 3x + 4$ is $\mathcal{O}(x)$. (Recall for positive-valued functions that f is $\mathcal{O}(g)$ if there are constants C and k such that $f(x) < Cg(x)$ for $x > k$.)

Our $g(x)$ is x, so we need to find C and k such that $3x+4 < Cx$ for $x > k$. So, C should be larger than 3; let's try $C = 4$. Then we need to find k so that $3x + 4 < 4x$ for $x > k$. But $3x + 4 < 4x$ when $4 < x$, so $k = 4$ works.

Besides the "witness" $(C, k) = (4, 4)$ found here, there are lots of other witnesses you could find.

Problem 4. [28; 4 points each part] True or false. Just write the word "true" or the word "false". If it's not clear to you which it is, explain; otherwise no explanation is necessary.

a. If A, B, and C are three sets, then the only way that $A \cup C$ can equal $B \cup C$ is if $A = B$. False. Take, for instance, when A and B are two different proper subsets of C.

b. There is no one-to-one correspondence between the set of all positive integers and the set of all odd positive integers because the second set is a proper subset of the first. False. There is one, namely, $f(n) = 2n - 1$.

c. If f is a function $A \to B$, and S and T are subsets of A, then $f(S \cap T) = f(S) \cap f(T)$. False. Take, for example, $f : \mathbf{Z} \to \mathbf{Z}$ where $f(n) = n^2$, $S = \{1, 2\}$, and $T = \{-1, -2\}$. Then $f(S \cap T) = \emptyset$ while $f(S) \cap f(T) = \{1, 4\}$.

d. $\lfloor 14.85 \rfloor + \lceil 14.85 \rceil = 30$. False. It's $14 + 15 = 29$.

e. $\sum_{k=0}^{10} 7 = 70$. False. There are 11 7s, so the sum is 77.

f. If $f(n) = (3n^6+5n^2-6)(n+\log n)$, then f is $\mathcal{O}(n^6 \log n)$. False. The order of the first term is n^6 while the order of the second is n, therefore the order of the product is n^7.

g. If the product $A \times B$ of two sets A and B is the emptyset $\emptyset$, then both A and B have to be $\emptyset$. False. If either one is empty, then the product will be, too.

Problem 5. On divisibility. [20] One of the theorems used to prove that Euclid's algorithm is valid is the following:

Theorem. The greatest common divisor of two positive integers a and b, where $a < b$, equals the greatest common divisor of a and $b - a$.

Prove this theorem that $\text{GCD}(a, b) = \text{GCD}(a, b - a)$. You can use the definition of $\text{GCD}(a, b)$ (the largest integer d that divides both a and b) and the definition of divides ($a \mid b$ iff $\exists c, ac = b$).

Of course, there are many possible proofs. All of them have in them somewhere that a divisor of two integers also divides both their sum and difference. Those statements could be proved first as lemmas making the proof of the theorem simpler, but there's not much to prove them and they can easily be incorporated in the example proof given here.

Proof. First, we'll show that set of common divisors of a and b is the same as the set of common divisors of a and $b - a$. It follows that the greatest element in this set is the GCD of both (a, b) and of $(a, b - a)$.

Suppose d is a common divisor of a and b, that is, $d \mid a$ and $d \mid b$. Then $\exists e$ such that $de = a$, and $\exists f$ such that $df = b$. Therefore, $b - a = df - dc = d(f - e)$. Hence, $d \mid (b - a)$. Thus the common divisors of (a, b) are also common divisors of $(a, b - a)$.

On the other hand, suppose d is a common divisor of a and $b - a$, that is, $d \mid a$ and $d \mid (b - a)$. Then $\exists e$ such that $de = a$, and $\exists f$ such that $df = b - a$. Therefore, $b = a + (b - a) = df + de = d(f + e)$. Hence, $d \mid b$. Thus the common divisors of $(a, b - a)$ are also common divisors of (a, b). Q.E.D.