Math 114 Discrete Mathematics

Second Midterm Anaswers

April 2018

Scale. 87–102 A, 75–86 B, 58–74 C. Median 94.

1. [15; 5 points each part] Give a big-$\mathcal{O}$ estimate for each of these functions. For the function g in your estimate that $f(x)$ is $\mathcal{O}(g)$, use a simple function g of smallest order. (You don't need to prove that your answer is correct.)

a. $\dfrac{x^4+\log x}{2x^2+3x+1}$

Since $\log x \prec x^4$, you may ignore $\log x$. Also since $3x+1 \prec x^2$, you may ignore $3x+1$. Also, the factor of 2 in $2x^2$ may be ignored. That leaves $\dfrac{x^4}{x^2}=x^2$. So this function is $\mathcal{O}(x^2)$.

b. $(x\log x+x^2)(x^3+2)$

Since $\log x \prec x$, therefore $x\log x \prec x^2$, so that term can be ignored. Also, the constant 2 can be ignored. That leaves $x^2x^3=x^5$. So this function is $\mathcal{O}(x^5)$.

c. $x^x+2^x+x!$

Since $2^x \prec x! \prec x^x$, this function is $\mathcal{O}(x^x)$.

2. [15] Use the definition of divisibility to prove that if a divides b, and c divides d, then ac divides bd. Assume that a, b, c, and d are all positive integers.

The definition for one integer a dividing another integer b, written $a\big|b$, is that there is a third integer c such that $ac=b$. Equivalently, $\dfrac{b}{a}$ is an integer.

If $a\big|b$ and $c\big|d$, then both $\dfrac{b}{a}$ and $\dfrac{d}{c}$ are integers. Therefore, their product $\dfrac{b}{a}\dfrac{d}{c}$ is also an integer. But that's equal to $\dfrac{bd}{ac}$. Thus, $ac\big|bd$.

3. [10; 5 points each part] Evaluate the following quantities.

a. 100 mod 7

7 goes into 100 20 times with a remainder of 2. So 100 mod $7=2$.

b. -3 mod 5

Here the question is, what's the smallest nonnegative integer less than 5 which is congruent to -3 modulo 5? If you add 5 to -3, you get 2, so -3 mod $5=2$.

4. [10] Use the Euclidean algorithm to find the greatest common divisor of 1039 and 323. (Show at least a couple of the intermediate steps.)

323 goes into 1039 3 times with a remainder of 70.
70 goes into 323 4 times with a remainder of 43.
43 goes into 70 once with a remainder of 27.
27 goes into 43 once with a remainder of 16.
16 goes into 27 once with a remainder of 11.
11 goes into 16 once with a remainder of 5.
5 goes into 11 twice with a remainder of 1.
Therefore, the $\textsc{gcd}(1039,323)=1$.

5. [12; 6 points each part] For this problem, you may leave your answers as algebraic expressions.

There are various ways to derive the answer. I'll give one. Also, the answers can often be simplified, but that's not required.

a. How many strings are there of lowercase letters of length four or less?

There are 26^4 strings of length 4, 26^3 strings of length 3,26^2 strings of length 2, 26^1 strings of length 1, and, if you consider the empty string to be a string, there's 26^0 strings of length 0. So $26^4+26^3+26^2+26^1+26^0$ strings in all.

b. How many strings are there of lowercase letters of length four or less that have the letter x in them?

There are 25^4 strings of length 4 that don't have x, so 26^4-25^4 that do have x. There are also shorter strings that don't have x, so the answer is $(26^4-25^4)+(26^3-25^3)+(26^2-25^2)+(26^1-25^1)$.

6. [20] On mathematical induction.

In this problem, you'll prove that

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

for positive integers n.

a. [2] Prove the base case.

The base case is when $n=1$, and that says $\dfrac{1}{1\cdot 3}=\dfrac{1}{2\cdot 1+1}$ which is true since both sides equal $\frac{1}{3}$.

b. [2] State the inductive hypothesis when $n=k$.

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

c. [2] State the inductive conclusion when $n=k+1$.

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

e. [14] Prove the inductive step that $n=k$ implies $n=k+1$.

By the inductive hypothesis, the first k terms of the left side of the inductive conclusion equal $\dfrac{k}{2k+1}$, so in order to show the inductive conclusion it's enough to show that

$\dfrac{k}{2k+1}$ plus the $k+1^{\text{st}}$ term equals the right side of the inductive conclusion:

$$\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}.$$

Simplifying the left side of that equation we have

$$\begin{aligned}
\frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} &= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\
&= \frac{k+1}{2k+3}
\end{aligned}$$

Therefore, the inductive conclusion holds.

7. [20; 5 points each part] For this problem, you may leave your answers as expressions in terms of factorials and binomial coefficients.

There are various ways to derive the answer. I'll give one. Also, the answers can often be simplified, but that's not required.

A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes

a. are there in total?

Each flip either gives heads or tails, and there are 8 flips, so by the multiplicative principle, there are 2^8 possible outcomes.

b. contain exactly three heads?

Three of the eight flips are heads, so there are 8 choose 3 outcomes of this kind. $\binom{8}{3}$.

c. contain at least three heads?

$$\binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8}$$

d. contain the same number of heads and tails?

This is asking how many outcomes contain exactly four heads. $\binom{8}{4}$.