Section 1.4, selected answers
Math 114 Discrete Mathematics
D Joyce, Spring 2018

4. $P(x, y)$ is "x has taken class y." Express the following in English. There are minor variants that you can give for each of these. Let's write Pxy rather than $P(x, y)$ whenever we can to reduce parentheses.

a. $\exists x\, \exists y\, Pxy$. Someone has taken some class.

b. $\exists x\, \forall y\, Pxy$. Someone has taken every class.

c. $\forall x\, \exists y\, Pxy$. Everyone has taken at least one class. Or "taken a class" or "taken some class."

d. $\exists y\, \forall x\, Pxy$. There's a class that everyone has taken.

e. $\forall y\, \exists x\, Pxy$. Every class has at least one student.

f. $\forall x\, \forall y\, Pxy$. Everyone has taken every class.

6. Let $C(x, y)$ mean that a student x is enrolled in a class y. Express the following in English. Of course, there are many variant sentences you can give. These are just samples.

a. $C(\text{Randy Goldberg}, \text{CS 252})$. Randy Goldberg is enrolled in CS 252.

b. $\exists x\, C(x, \text{Math 695})$. Someone is taking Math 695.

c. $\exists y\, C(\text{Carol Sitea}, y)$. Carol Sitea is enrolled in some class.

d. $\exists x\, C(x, \text{Math 222}) \wedge C(x, \text{Math 252})$. Somebody is taking both Math 222 and Math 252.

e. $\exists x\, \exists y\, \forall z\, ((x \neq y) \wedge (Cxz \to Cyz)$. There are two students such that the every class that the first student is taking is also taken by the second student. More briefly, but a little less clearly, you could say: somebody is enrolled in every class someone else is taking.

f. $\exists x\, \exists y\, \forall z\, ((x \neq y) \wedge (Cxz \leftrightarrow Cyz)$. Somebody is taking the exact same classes that somebody else is taking.

12. Let $I(x)$ be "x has an internet connection" and let $C(x, y)$ be "x and y have chatted over the internet." Assume the universe of discourse consists of all students in your class. Express the following using quantifiers.

An alternative way of writing predicates is to drop the parentheses, that is, write Ix instead of $I(x)$, and Cxy instead of $C(x, y)$. Some of these statements have so many parentheses that it makes sense to drop these extra parentheses, especially when the arguments are variables.

a. Jerry does not have an Internet connection. $\neg I(\text{Jerry})$.

b. Rachel has not chatted over the internet with Chelsea. $\neg C(\text{Rachel,Chelsea})$.

c. Jan and Sharon have never chatted over the internet. $\neg C(\text{Jan,Sharon})$.

d. No one in the class has chatted with Bob. $\neg\exists x\, C(x, \text{Bob})$. Alternatively, $\forall x\, \neg C(x, \text{Bob})$.

e. Sanjay has chatted with everyone except Joseph. $\forall y\, (y \neq \text{Joseph} \to C(\text{Sanjay}, y))$. Alternatively, $\forall y\, (y = \text{Joseph} \vee C(\text{Sanjay}, y))$.

f. Someone in your class does not have an internet connection. $\exists x\, \neg Ix$.

g. Not everyone in your class has an internet connection. $\neg\forall x\, Ix$. This is logically equivalent to the previous statement.

h. Exactly one student in your class has an internet connection. $\exists x\, (Ix \wedge \forall y\, (Iy \to y = x))$. Alternatively, $\exists x\, (Ix \wedge \forall y\, (y \neq x \to \neg Iy))$. Using the uniqueness quantifier, you could write this as $\exists!\, x\, Ix$.

i. Everyone except one student in your class has an internet connection. It appears this means exactly one student does not have a connection. That would be $\exists!\, x\, \neg Ix$ when expressed with the uniqueness quantifier, but without the uniquenss quantifier it could be $\exists x\, (\neg Ix \wedge \forall y\, (Iy \to y = x))$.

j. Everyone in your class with an internet connection has chatted over the internet with at least one other student in your class. $\forall x\, (Ix \to \exists y\, (Cxy \wedge x \neq y))$. If you assume nobody can chat with him or herself, a reasonable assumption, then you can drop the $x \neq y$ at the end.

k. Someone in your class has an internet connection but has not chatted with anyone else in your class. $\exists x\, (Ix \wedge \forall y\, \neg Cxy)$. Alternatively, $\exists x\, (Ix \wedge \neg\exists y\, Cxy)$.

l. There are two students in your class who have not chatted with each other over the internet. $\exists x\, \exists y\, (x \neq y \wedge \neg Cxy)$. Sometimes this would be abbreviated $\exists x\, \exists y \neq x, \neg Cxy$.

m. There is a student in your class who has chatted with everyone in your class over the internet. $\exists x\, \forall y\, Cxy$. It's reasonable to assume that the statement had an implicit "else" in it: ... with everyone else ..., in which case you need $\exists x\, \forall y\, (x \neq y \to Cxy)$.

n. There are at least two students in your class who have not chatted with the same person in your class. $\exists x\, \exists y\, (x \neq y \wedge \exists z\, (\neg Cxz \wedge \neg Cyz)$. Probably, the intent of the sentence is that z is a third person, so a better translation would be $\exists x\, \exists y\, (x \neq y \wedge \exists z\, (x \neq z \wedge y \neq z \wedge \neg Cxz \wedge \neg Cyz)$.

o. There are two students in the class who between them have chatted with everyone else in the class. $\exists x\, \exists y\, (x \neq y \wedge \forall z\, (Cxz \vee Cyz))$.

20. Express these statements using predicates, quantifiers, logical connectives, and mathematical operators where the universe of discourse is the set of all integers.

a. The product of two negative integers is positive.

One logical expression that does the job is

$$\forall x, \forall y,\ (x < 0 \wedge y < 0 \to xy > 0).$$

But we usually combine the condition $x < 0$ with the quantifier to make a shorter expression:

$$\forall x < 0, \forall y < 0,\ xy > 0.$$

b. The average of two positive integers is positive.

$$\forall x > 0, \forall y > 0,\ \frac{x + y}{2} > 0.$$

c. The difference of two negative integers is not necessarily negative.

You could say it is not the case that the difference is always negative as follows

$$\neg\forall x < 0, \forall y < 0,\ x - y < 0,$$

or you could say that there are negative numbers whose difference is not negative:

$$\exists x < 0, \exists y < 0,\ x - y \not< 0.$$

The two statements are logically equivalent. Note that you could write $\geq$ rather than $\not<$.

d. The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

$$\forall x, \forall y,\ |x + y| \leq |x| + |y|.$$

This inequality is sometimes called the *triangle inequality.*

28. Determine the truth value . The universe of discourse is the set of real numbers. Think of these as questions.

a. $\forall x\, \exists y\, (x^2 = y)$. For each x is there a y such that $x^2 = y$? Yes, x^2.

b. $\forall x\, \exists y\, (x = y^2)$. Is every number x the square of some number y? No, not the negative ones; negative numbers don't have square roots.

c. $\exists x\, \forall y\, (xy = 0)$. Is there some number whose product with any other number is 0? Yes there is, namely 0.

d. $\exists x\, \exists y\, (x+y \neq y+x)$. Can you find two numbers whose sum depends on the order you add them? No. Addition is commutative.

e. $\forall x{\neq}0\ \exists y\, (xy = 1)$. This is an abbreviation for $\forall x\, (x \neq 0 \to \exists y\, (xy = 1)$. For any nonzero real number x can you solve $xy = 1$ for y? Yes, $y = 1/x$.

f. $\exists x\, \forall y{\neq}0\, (xy = 1)$. Compare this with part d above. Is there some number whose product with any other number is 1? No.

g. $\forall x\, \exists y\, (x + y = 1)$. Can you solve $x + y = 1$ for y in terms of x? Yes, $y = 1 - x$.

h. $\exists x\, \exists y\, (x + 2y = 2 \wedge 2x + 4y = 5)$. Can you solve the pair of simultaneous equations $x + 2y = 2$ and $2x + 4y = 5$ for x and y? No, there are no solutions. If you double the first you get $2x + 4y = 4$, but the second says $2x + 4y = 5$. Since $4 \neq 5$, there are no solutions.

i. $\forall x\, \exists y\, (x + y = 2\ \wedge\ 2x - y = 1)$. For any x can you find a y that satisfies both $x + y = 2$ and $2x - y = 1$? According to the first equation, $y = 2 - x$, but according to the second equation, $y = 2x - 1$. Since $2 - x$ does not equal $2x - 1$ for all x, both equations can't be satisfied for all x.

j. $\forall x\, \forall y\, \exists z \left(z = \dfrac{x+y}{2}\right)$. Yes. Given any two numbers x and y, you can indeed average them.

32. Express the negations of each of these statements so that all negation symbols immediately precede predicates.

As you pass the $\neg$ past a quantifier, exchange $\forall$ and $\exists$, and as you pass it past either $\wedge$ or $\vee$, exchange them.

a. $\forall z\, \exists y\, \exists x\, \neg T(x, y, z)$

b. $\forall x\, \forall y\, \neg P(x, y) \vee \exists x\, \exists y\, \neg Q(x, y)$

c. $\forall x\, \forall y\, (Q(x, y) \oplus Q(y, x))$, or $\forall x\, \forall y\, (\neg Q(x, y) \leftrightarrow Q(y, x))$, or $\forall x\, \forall y\, (Q(x, y) \leftrightarrow \neg Q(y, x))$.

d. $\exists y\, \forall x\, \forall z\, (\neg T(x, y, z) \wedge \neg Q(x, y))$

46. Determine the truth value of the statement

$$\exists x\, \forall y\, (x \leq y^2)$$

if the universe of discourse for the variables consists of

a. the positive real numbers. How about this? Just let x be 0. The square y^2 of any real number y is greater than or equal to 0. Wait a minute. We're not allowed to let x be 0 since x has to be positive. Then it's false. No matter what positive number x is, there are squares y^2 that are smaller than x.

b. the integers. Yep, $x = 0$ is an integer that works.

c. the nonzero real numbers. We're not allowed to let x be 0. But we can take x to be negative, say $x = -1$. Then every square y^2 is greater than x.

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