Section 3.2, selected answers
Math 114 Discrete Mathematics
D Joyce, Spring 2018

2. Determine whether each of these functions is \( O(x^2) \).
   a. \( f(x) = 17x + 11 \). Yes. By theorem 1, any linear function is \( O(x) \), and since any \( O(x) \) function is also \( O(x^2) \), this function is \( O(x^2) \).
   b. \( f(x) = x^2 + 1000 \). Yes. By theorem 1, any quadratic function is \( O(x^2) \).
   c. \( f(x) = x \log x \). Yes. We know \( x \) is \( O(x) \). We also know \( \log x \) is \( O(x) \). Therefore, their product is \( O(x^2) \).
   d. \( f(x) = x^4/2 \). Theorem 4 is a strengthening of theorem 1. It implies this degree 4 polynomial of \( O(x^4) \), but it not \( O(x^n) \) for any \( n < 4 \). Therefore, this function is not \( O(x^2) \).
   e. \( f(x) = 2^x \). This exponential function is not \( O(x^2) \).
   f. \( f(x) = \lfloor x \rfloor \cdot \lceil x \rceil \). Both floor and ceiling are \( O(x) \), so their product is \( O(x^2) \).

8. Find the least integer \( n \) such that \( f(x) \) is \( O(x^n) \) for each of these functions.
   a. \( f(x) = 2x^2 + x^3 \log x \). Since \( x^3 \) dominates \( x^2 \), we can ignore the first term and concentrate on the second term, \( x^3 \log x \). Since \( x^3 \log x \) is \( O(x^4) \), but it’s not \( O(x^3) \), the \( n \) we’re looking for here is 4.
   b. \( f(x) = 3x^5 + (\log x)^4 \). Any positive power of \( x \) dominates any power of \( \log x \), so we can ignore the second term. Then, \( f \) is \( O(x^5) \).
   c. \( f(x) = (x^4 + x^2 + 1)/(x^4 + 1) \). Divide the denominator into the numerator in order to write the function as

\[
  f(x) = 1 + \frac{x^2}{x^4 + 1}.
\]

Since the fraction is \( O(1) \), therefore \( f(x) \) is \( O(1) \).
   d. \( f(x) = (x^3 + 5 \log x)/(x^4 + 1) \). The denominator is bigger than the numerator! Since \( x^4 \) dominates 1, and \( x^3 \) dominates \( \log x \), we can disregard those terms and reduce the problem to finding the order of \( x^3/x^4 \), but that’s just \( x^{-1} \). Thus \( f(x) \) is \( O(x^{-1}) \), and the \( n \) we’re looking for is \(-1\).

20. Find the order of these functions.
   a. \( (n^3+n^2 \log n)(\log n+1)+(17 \log n+19)(n^3+2) \). Whenever you’ve got the sum of two terms, ignore the smaller one. Then this expression simplifies to \( n^3 \log n + 17n^2 \log n \). We can drop the constant 17. Also, since \( \log n \) is \( O(n) \), therefore \( n^2 \log n \) is \( O(n^3) \). Thus, this function is on the order of \( n^3 \log n \).
   b. \( (2^n + n^2)(n^3 + 3^n) \). Since \( 2^n \) dominates \( n^2 \), and \( 3^n \) dominates \( n^3 \), this is on the order of \( 2^n \cdot 3^n \), which is \( 6^n \).
   c. \( (n^n + n2^n + 5^n)(n! + 5^n) \). The function \( n^n \) dominates both \( 2n^n \) and \( 5^n \), while \( n! \) dominates \( 5^n \), so this function is on the order of \( n^n \cdot n! \).