Section 3.2, selected answers
Math 114 Discrete Mathematics
D Joyce, Spring 2018

2. Determine whether each of these functions is $\mathcal{O}(x^2)$.

a. $f(x) = 17x + 11$. Yes. By theorem 1, any linear function is $\mathcal{O}(x)$, and since any $\mathcal{O}(x)$ function is also $\mathcal{O}(x^2)$, this function is $\mathcal{O}(x^2)$.

b. $f(x) = x^2 + 1000$. Yes. By theorem 1, any quadratic function is $\mathcal{O}(x^2)$.

c. $f(x) = x \log x$. Yes. We know x is $\mathcal{O}(x)$. We also know $\log x$ is $\mathcal{O}(x)$. Therefore, their product is $\mathcal{O}(x^2)$.

d. $f(x) = x^4/2$. Theorem 4 is a strengthening of theorem 1. It implies this degree 4 polynomial of $\mathcal{O}(x^4)$, but it not $\mathcal{O}(x^n)$ for any $n < 4$. Therefore, this function is not $\mathcal{O}(x^2)$.

e. $f(x) = 2^x$. This exponential function is not $\mathcal{O}(x^2)$.

f. $f(x) = \lfloor x \rfloor \cdot \lceil x \rceil$. Both floor and ceiling are $\mathcal{O}(x)$, so their product is $\mathcal{O}(x^2)$.

8. Find the least integer n such that $f(x)$ is $\mathcal{O}(x^n)$ for each of these functions.

a. $f(x) = 2x^2 + x^3 \log x$. Since x^3 dominates x^2, we can ignore the first term and concentrate on the second term, $x^3 \log x$. Since $x^3 \log x$ is $\mathcal{O}(x^4)$, but it's not $\mathcal{O}(x^3)$, the n we're looking for here is 4.

b. $f(x) = 3x^5 + (\log x)^4$. Any positive power of x dominates any power of $\log x$, so we can ignore the second term. Then, f is $\mathcal{O}(x^5)$.

c. $f(x) = (x^4 + x^2 + 1)/(x^4 + 1)$. Divide the denominator into the numerator in order to write the function as

$$f(x) = 1 + \frac{x^2}{x^4 + 1}.$$

Since the fraction is $\mathcal{O}(1)$, therefore $f(x)$ is $\mathcal{O}(1)$.

d. $f(x) = (x^3 + 5 \log x)/(x^4 + 1)$. The denominator is bigger than the numerator! Since x^4 dominates 1, and x^3 dominates $\log x$, we can disregard those terms and reduce the problem to finding the order of x^3/x^4, but that's just x^{-1}. Thus $f(x)$ is $\mathcal{O}(x^{-1})$, and the n we're looking for is -1.

20. Find the order of these functions.

a. $(n^3+n^2 \log n)(\log n+1)+(17 \log n+19)(n^3+2)$. Whenever you've got the sum of two terms, ignore the smaller one. Then this expression simplifies to $n^3 \log n + 17n^2 \log n$. We can drop the constant 17. Also, since $\log n$ is $\mathcal{O}(n)$, therefore $n^2 \log n$ is $\mathcal{O}(n^3)$. Thus, this function is on the order of $n^3 \log n$.

b. $(2^n + n^2)(n^3 + 3^n)$. Since 2^n dominates n^2, and 3^n dominates n^3, this is on the order of $2^n \cdot 3^n$, which is 6^n.

c. $(n^n + n2^n + 5^n)(n! + 5^n)$. The function n^n dominates both $n2^n$ and 5^n, while $n!$ dominates 5^n, so this function is on the order of $n^n \cdot n!$

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