9. Find a formula for the sum of the first $n$ even positive integers, and use mathematical induction to prove your formula is correct.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2n$</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>$\sum$</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>20</td>
<td>30</td>
<td>42</td>
</tr>
</tbody>
</table>

It appears that the sum $\sum_{k=1}^{n} 2k$ is $n^2 + n$.

To prove it using math induction, first check the base case. Is the sum of the first $n = 1$ even number (which is just 2) equal to $1^2 + 1$? Yes, it is.

Next, the inductive step. Assume the inductive hypothesis $P(n)$:

$$\sum_{k=1}^{n} 2k = n^2 + n$$

is true, and prove that $P(n + 1)$:

$$\sum_{k=1}^{n+1} 2k = (n + 1)^2 + (n + 1)$$

is also true. Starting with the LHS

$$\sum_{k=1}^{n+1} 2k = \left(\sum_{k=1}^{n} 2k\right) + 2(n + 1)$$

which, by the inductive hypothesis,

$$= (n^2 + n) + 2(n + 1)$$

With a little algebra, that can be written as the RHS:

$$= n^2 + 2n + 1 + n + 1 = (n + 1)^2 + (n + 1).$$

Therefore, the inductive step is valid and the proof is complete.

10. Find a formula for

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)}$$

by examining the values of this expression for small values of $n$. Use mathematical induction to prove your result.

Start by doing the additions, not with a calculator, but by hand. If you use a calculator, you probably won’t see the pattern.

$$\frac{1}{1 \cdot 2} = \frac{1}{2}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{2}{3}$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{3}{4}$$

Aha! The pattern suggests that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}.$$

Now, to prove it. The first line shows it’s true for the base case.

Next, the inductive step. Assume the inductive hypothesis

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}$$

and prove that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n + 1)} + \frac{1}{n + 1} = \frac{n + 1}{n + 2}.$$

By the inductive hypothesis, the LHS

$$= \frac{n}{n + 1} + \frac{1}{(n + 1)(n + 2)}.$$

But that simplifies to

$$= \frac{n(n + 2) + 1}{(n + 1)(n + 2)} = \frac{n^2 + 2n + 1}{(n + 1)(n + 2)} = \frac{(n + 1)^2}{(n + 1)(n + 2)},$$

and that reduces to the RHS.

Therefore, the inductive step is valid, and the proof is complete.
14. Prove that for every positive integer \( n \),
\[
\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2.
\]

The base is when \( n = 1 \). In that case, the equation says that \( 1 \cdot 2^1 = (1-1)2^{1+1} + 2 \), which is true.

For the inductive step, we assume the inductive hypothesis that for a given value of \( n \),
\[
\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2,
\]
and we have to show that for the next value of \( n \), namely \( n + 1 \), that
\[
\sum_{k=1}^{n+1} k2^k = ((n + 1) - 1)2^{(n+1)+1} + 2,
\]
that is, that
\[
\sum_{k=1}^{n+1} k2^k = n2^{n+2} + 2.
\]

16. Use mathematical induction to prove that
\[
1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = (n+1)(n+2)(n+3)/4.
\]

Let’s denote the left hand side of this equation by LHS(\( n \)) and the right hand side by RHS(\( n \)).

You have to decide what the base case is. Is it \( n = 0 \) or \( n = 1 \)? It isn’t clear from the statement of the problem. If you take \( n = 0 \) to be the base case, then LHS(0) is an empty sum, and an empty sum is 0. But RHS(0) is \( 0 \cdot 1 \cdot 2 \cdot 3/4 \), which is also 0. The other choice for a base case is \( n = 1 \). Then LHS(1) is \( 1 \cdot 2 \cdot 3 \), while RHS(1) \( 1 \cdot 2 \cdot 3 \cdot 4/4 \), so the two sides are equal. So, the base case is taken care of either with \( n = 0 \) or with \( n = 1 \).

Now for the inductive step. Suppose that the statement is true for \( n \), that is, LHS(\( n \)) = RHS(\( n \)). We have to prove that it’s true for \( n + 1 \), that is, LHS(\( n + 1 \)) = RHS(\( n + 1 \)), which written is full is
\[
1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + (n + 1)(n + 2)(n+2) = (n+1)(n+2)(n+3)(n+4)/4.
\]

The only difference between the LHS(\( n \)) and LHS(\( n + 1 \)) is that the latter has one more term, namely, \((n+1)(n+2)(n+2)\). That is,
\[
\text{LHS}(n + 1) - \text{LHS}(n) = (n+1)(n+2)(n+2).
\]

If we can show that RHS(\( n + 1 \)) - RHS(\( n \)) has the same value, then we may conclude LHS(\( n \)) = RHS(\( n \)) implies LHS(\( n + 1 \)) = RHS(\( n + 1 \)). But
\[
\text{RHS}(n+1) - \text{RHS}(n) = (n+1)(n+2)(n+3)(n+4)/4 - n(n+1)(n+2)(n+3)/4
\]
\[
= ((n+4) - n)(n+1)(n+2)(n+3)/4
\]
\[
= (n+1)(n+2)(n+3)
\]
That finishes the inductive step, so we’ve finished the proof.

21. Show that \( 2^n > n^2 \) whenever \( n \) is an integer greater than 4.

In this case, the base case occurs when \( n = 5 \), so you need to check that \( 2^5 > 5^2 \), which, of course, is true.

Now for the inductive step. Assume the inductive hypothesis \( 2^n > n^2 \), where \( n > 4 \), and prove that \( 2^{n+1} > (n+1)^2 \).
\[
2^{n+1} = 2 \cdot 2^n > 2n^2
\]
At this point, it suffices to show that \( 2n^2 > (n+1)^2 \), and that’s logically equivalent to the inequality \( n^2 > 2n+1 \), or \( n^2 - 2n+1 > 2 \), that is, \((n-1)^2 > 2 \). But \( n > 4 \), so \((n-1)^2 > 9 > 2 \). Thus, the inductive conclusion \( 2^{n+1} > (n+1)^2 \) follows, and the proof is complete.