Math 114 Discrete Mathematics

Section 5.1, selected answers

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Remember that for problems like this, the important thing is understanding how to derive the answer; the answer itself is usually irrelevant. So make sure you explain how you derive the answer.

7. How many different three-letter initials can people have?

That would be 26^3. I wonder who has initials X.X.X.?

8. How many initials where none of the letters are repeated?

That's $26 \cdot 25 \cdot 24$, which can also be written using factorials as $26!/23!$.

10. How many bit strings are there of length 8?

There are 2^8 which is 256. That means there are 256 different values you can store in a byte, since a byte is eight bits. There are 256 eight-bit ascii codes, for instance.

11. How many bit strings are there of length 10 begin and end with a 1?

The answer is the same as the answer in exercise 10 since only the middle 8 bits can be either 0 or 1.

12. How many bit strings are there of length 6 or less?

It's the "or less" that makes this an interesting problem. There are 2^6 strings of length 6; 2^5 of length 5; etc. down to 2^0 strings of length 0 (that's the empty string). So, altogether, that gives

$$2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2 + 1 = 2^7 - 1 = 127$$

bit strings altogether.

16. How many strings are there of four lowercase letters that have the letter `x` in them?

There are several ways to find the number.

Here's a difficult way that requires the Principle of Inclusion and Exclusion. There are four places that the `x` can be: the first letter, the second, the third, or the fourth. Let S_1 be the set of strings of length 4 with an `x` in the first place, and analogously for S_2, S_3, and S_4. The size of S_1 is 26^3 since the other three places can have any letter. Likewise, $|S_2| = |S_3| = |S_4| = 26^3$. But $|S_1| + |S_2| + |S_3| + |S_4|$ does not equal $|S_1 \cup S_2 \cup S_3 \cup S_4|$ since if an element is in two of the subsets (that is, the string has two `x`'s) it would be counted twice. So the sizes of all combinations of pairwise intersections have to be subtracted, such as $|S_1 \cap S_2|$. But then you have to add back the sizes of triple intersections like $|S_1 \cap S_2 \cap S_3|$. And you'll have to subtract the size of the quadruple intersection $|S_1 \cap S_2 \cap S_3 \cap S_4|$. If you all do that, it'll work, but there's an easier way.

First count the number of strings of any four letters. That's 26^4. Then count the number of strings that don't have any `x`. That'll be 25^4 since each of the letters can be any of 25 possibilities. A string will have at least one `x` if it's one of the 26^4 strings of length 4, but not one of the 25^4 strings that don't have an `x` in them. Thus, the answer is $26^4 - 25^4$.

23. How many strings are there of three decimal digits that

a. do not contain the same digit three times?

First, there are 1000 strings of length 3 made out of the ten digits. But 10 of them are triple repeats like 666. Excluding those 10 leaves 990.

b. begin with an odd digit?

The first digit can be any of 5 digits, but the other two can be any of 10 digits, so the answer is $5 \cdot 10 \cdot 10 = 500$.

c. have exactly two digits that are 4s?

If the first two digits are both 4, then the third can be any of 9 digits (not 4), similarly if the two 4s appear either in the first and third position or in the second and third position. So the answer is $9 + 9 + 9 = 27$.

24. How many strings of four decimal digits

a. do not contain the same digit twice?

This is the number of 4-permutations of a set of 10 elements: $10 \cdot 9 \cdot 8 \cdot 7$.

b. end with an even digit?

The first three can be any of 10 values, the last any of 5 values, so the answer is $10 \cdot 10 \cdot 10 \cdot 5$.

c. have exactly three digits which are 9s?

The fourth digit, which can be any of 9 possibilities, can appear in any one of 4 positions. That gives $9 \cdot 4 = 36$.

31. How many strings of eight English letters are there

a. that contain no vowels, if letters can be repeated?

Let's agree that there are 5 vowels (ignore `y`). Then any of 21 letters can appear in any of the 8 positions, so the answer is 21^8.

b. that contain no vowels, if letters cannot be repeated?

The first can be any of 21 letters, the next can't be the same as the first, so there are 20 possible letters, the third can be any of 19, etc. Therefore, the answer is $21 \cdot 20 \cdots 14$, that is $21!/13!$.

c. that start with a vowel if letters can be repeated?

That's $5 \cdot 26^7$ since there are 5 possibilities for the first letter and 26 for each of the rest.

d. that start with a vowel if letters cannot be repeated? There are 5 possibilities for the first letter, 25 for the second, 24 for the third, etc. That gives $5 \cdot 25 \cdot 24 \cdots 18 = 5 \cdot 25!/17!$.

e. that contain at least one vowel if letters can be repeated?

As in problem 16, it's a lot easier to subtract the number of strings that don't contain a vowel from the number of all the strings. There are 26^8 strings in all, but only 21^8 that don't have a vowel. Therefore, there are $26^8 - 21^8$ that have at least one vowel.

f. that contain exactly one vowel if letters can be repeated?

The vowel (any of 5 possibilities) can occur in any of 8 places, and the other 7 places each can have any of 21 possible letters. That gives $5 \cdot 8 \cdot 21^7$.

g. that start with an `X` and contain at least one vowel, if letters can be repeated?

There are 26^7 strings that start with an `X`, and 21^7 that don't have any vowel, so the answer is $26^7 - 21^7$.

h. that start and end with an `X` and contain at least one vowel, if letters can be repeated?

There are 26^6 strings that start and end with an `X`, and 21^6 of those that don't have any vowel, so the answer is $26^6 - 21^6$.

39. A *palindrome* is a string whose reversal is identical to the string, such as `noon` and `madam`. How many bit strings of length n are palindromes?

It depends on if n is even or odd. If n is even, then the first $n/2$ can be any letter, and the rest are determined, so in that case, there are $26^{n/2}$. But if n is odd, then the first $\lceil n/2 \rceil$ can be any letter (including the one in the middle), and the remaining are determined, so in that case, there are $26^{\lceil n/2 \rceil}$.

(In the even case, since $n/2$ does equal $\lceil n/2 \rceil$, we can join the two cases together to conclude the number is $26^{\lceil n/2 \rceil}$.)

42. How many bit strings of length 7 either begin with two 0s or end with three 1s?

This is best done with PIE (principle of inclusion and exclusion). The number of bit strings of length 7 either begin with two 0's or end with three 1s is the number that begin with two 0s plus the number that end with three 1s minus the number that both begin with two 0s and end with three 1s. That gives $2^5 + 2^4 - 2^2$.

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