Math 114 Discrete Mathematics

Section 6.2, selected answers

D Joyce, Spring 2018

7. What is the probability of these events when we randomly select a permutation of the set $\{1, 2, 3, 4\}$.

a. 1 precedes 4.

There are $4! = 24$ permutations of the set $\{1, 2, 3, 4\}$. How many of them put 1 before 4? You can find many ways to count them.

Here's a tedious way to do that. You could put 1 first, then there are 6 ways to put 2, 3, and 4 after the 1. You could put 1 second, then there are 2 ways of putting 4 after 1 and then 2 ways of putting 2 and 3 in the remaining slots, giving 4 ways of putting 1 second. You could put 1 third, then put 4 fourth, and there are 2 ways of putting 2 and 3 in the remaining slots. You can't put 1 fourth. That gives $6 + 4 + 2 = 12$ ways of putting 1 before 4. That gives a probability of $\frac{12}{24} = \frac{1}{2}$.

But there's a much easier way to find the answer. Half the permutations have 1 before the 4, and half have 1 after the 4. A bijection between the two sets of permutations is effected by exchanging the 1 and 4 in a permutation.

b. 4 precedes 1.

By symmetry, the answer is the same as for part a, namely $\frac{1}{2}$.

c. 4 precedes both 1 and 2.

A similar argument shows there are 8 permutations that have 4 before both 1 and 2, and that gives a probability of $\frac{8}{24} = \frac{1}{3}$.

d. 4 precedes all three of 1, 2, and 3.

Put 4 first, then there are $3! = 6$ ways to place the other three. That gives a probability of $\frac{6}{24} = \frac{1}{4}$.

e. 4 precedes 3 and 2 precedes 1.

Of the $4! = 24$ permutations of $\{1, 2, 3, 4\}$, some have 4 before 3 and 2 before 1. They are 4321, 4231, 4213, 2431, 2413, and 2143. Since there are 6 of them, the probability is $\frac{6}{24} = \frac{1}{4}$.

8. What is the probability of these events when we randomly select a permutation of the set $\{1, 2, \ldots, n\}$ where $n \geq 4$?

a-b. 1 precedes 2, or 2 precedes 1.

The answers to part a and part b are each $\frac{1}{2}$ since either 1 precedes 2 or vice versa, and by symmetry they have the same probability.

c. 1 immediately precedes 2.

This is a harder question. How many of the $n!$ permutations have 1 immediately preceding 2? Here's one way of counting them. Treat the pair $(1, 2)$ as an individual element, and ask how many ways it can be permuted with the other $n - 2$ elements. There are $(n - 1)!$ permutations. So of the $n!$ permutations of $\{1, 2, \ldots, n\}$, $(n - 1)!$ have 1 immediately preceding 2. Thus, the probability is $(n - 1)!/n!$, which is $1/n$.

d. n precedes 1 and $n - 1$ precedes 2.

There are many ways to approach this question. Here's one that uses independence of events. The probability that n precedes 1 is $\frac{1}{2}$, and the probability that $n-1$ precedes 2 is also $\frac{1}{2}$. Since whether or not n precedes 1 has nothing whatever to do with $n-1$ preceding 2, the events are independent, and so the probabilities multiply. Therefore, the probability is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

e. n precedes 1 and n precedes 2.

We may assume $n = 3$ since the placements of the remaining elements are irrelevant. Of the 6 permutations of $\{1, 2, 3\}$, only 2 have 3 before both 1 and 2, so the probability is $\frac{2}{6} = \frac{1}{3}$.

12. Suppose that E and F are events such that $p(E) = 0.8$ and $p(F) = 0.6$. Show that

$$p(E \cap F) \geq 0.4.$$

The principle of inclusion and exclusion, P.I.E., says

$$p(E \cup F) = p(E) + p(F) - p(E \cap F).$$

Therefore,

$$p(E \cup F) = 1.4 - p(E \cap F).$$

But $p(E \cup F) \leq 1$, so

$$1 \geq 1.4 - p(E \cap F).$$

Therefore, $p(E \cap F) \geq 1.4 - 1 = 0.4$.

17. If E and F are independent events, prove or disprove that $\overline{E}$ and F are necessarily independent events.

Suppose that E and F are independent events. Then $p(E \cap F) = p(E)p(F)$. Now,

$$\begin{aligned} p(\overline{E})p(F) &= (1 - p(E))p(F) \\ &= p(F) - p(E)p(F) \\ &= p(F) - p(E \cap F). \end{aligned}$$

Next note that since F is the disjoint union of $E \cap F$ and $\overline{E} \cap F$. Therefore,

$$p(F) = p(E \cap F) + p(\overline{E} \cap F).$$

That implies

$$p(F) - p(E \cap F) = p(\overline{E} \cap F).$$

Thus, we've shown that $p(\overline{E})p(F) = p(\overline{E} \cap F)$, so that $\overline{E}$ and F are independent events.

25. What's the conditional probability that a randomly generated bit string of length 4 contains at least two consecutive 0s, given that the first bit is a 1?

You could note that this conditional prob. is the same as the prob. that a string of length 3 has at least two consecutive 0s. There are three ways to do that (000, 001, and 100) out of 8 possible strings giving a probability of $\frac{3}{8}$.

26. Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with a 1. Are E and F independent?

$E = \{100, 010, 001, 111\}$.
$F = \{100, 101, 110, 111\}$.
$E \cap F = \{100, 111\}$.

Since $p(E \cap F) = \frac{1}{4}$, and $p(E)p(F) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, therefore E and F *are* independent events.

28. Assume that the prob. a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the prob. that a family of 5 children has

a. exactly 3 boys?

There are $\binom{5}{3} = 10$ possible arrangements of 3 boys among 5 children. Each arrangement has probability $0.51^3 0.49^2$. Therefore the probability of this event is $10 \cdot 0.51^3 0.49^2$, which is approximately 0.318.

b. at least one boy?

All but one of the 32 possibilities have at least one boy, and that one occurs with probability 0.49^5. So the prob. of at least one boy is $1 - 0.49^5$, which is approximately 0.972.

c. at least one girl?

That's $1 - 0.51^5$, approximately 0.965.

d. all children of the same sex?

There are two outcomes in this event, one with prob. 0.51^5, the other with 0.49^5. Their sum is $0.51^5 + 0.49^5$, approximately 0.0627.

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