Math 114 Discrete Mathematics

Section 8.5, selected answers

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1. Which of these relations on the set $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a. $\{(0,0),(1,1),(2,2),(3,3)\}$.

It is an equivalence relation. In fact, it's equality, the best equivalence relation.

b. $\{(0,0),(0,2),(2,0),(2,2),(2,3),(3,2),(3,3)\}$.

It's not reflexive because it doesn't include $(1,1)$. It is symmetric. It's not transitive since $(0,2)$ and $(2,3)$ but not $(0,3)$.

c. $\{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3)\}$.

It is an equivalence relation.

d. $\{(0,0),(1,1),(1,3),(2,2),(2,3),(3,1),(3,2),(3,3)\}$.

It's reflexive and symmetric. It's not transitive since $(1,2)$ is missing.

e. $\{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,2),(3,3)\}$.

It's reflexive, but not symmetric or transitive. $(0,2)$ is missing.

2. Which of these relations on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

a. "a and b are the same age."

This is an equivalence relation. Any relation that can be expressed using "have the same" are "are the same" is an equivalence relation.

b. "a and b have the same parents."

That's an equivalence relation, too.

c. "a and b share a common parent."

This relation is reflexive and symmetric, but not transitive. It may be that half-siblings a and b have the same father, and half-siblings b and c have the same mother, but a and c are unrelated.

d. "a and b have met."

You can interpret this so that it's reflexive if you agree that everyone has automatically met him/herself. In any case it is symmetric. It's not transitive.

e. "a and b speak a common language."

Again, this is reflexive and symmetric, but not transitive.

3. Which of the following relations on the set of all functions from $\mathbf{Z}$ to$\mathbf{Z}$ are equavalence relations?

a. $\{(f,g) \mid f(1) = g(1)\}$.

"Having the same value at 1" is an equivalence relation.

b. $\{(f,g) \mid f(0) = g(0) \text{ or } f(1) = g(1)\}$.

This isn't transitive.

c. $\{(f,g) \mid \forall x\ f(x) - g(x) = 1\}$.

Not reflexive since $f(x) - f(x) = 0$. Notsymmetric since if $f(x) - g(x) = 1$, then $g(x) - f(x) = -1$. Not transitive either, since if $f(x) - g(x) = 1$ and $g(x) - h(x) = 1$, then $f(x) - h(x) = 2$.

d. $\{(f,g) \mid \exists C\ \forall x\ f(x) - g(x) = C\}$.

"Differing by a constant" is an equivalence relation. The graphs of the two functions are the same, except for a vertical shift. (If the functions under consideration are all differentiable, then this says they have the same derivative.)

e. $\{(f,g) \mid f(0) = g(1) \text{ and } f(1) = g(0)\}$.

It isn't reflexive since $f(0) = f(1)$ isn't always true. Not transitive either. For example, let $f(x) = h(x) = x$ and $g(x) = 1 - x$. Then $f \equiv g$ and $g \equiv h$, but not $f \equiv h$.

21-23. Determine whether the relation given graphically is an equivalence relation.

For 21. No. Not transitive since $c \to a$ and $a \to d$, but not $c \to d$.

For 22. Yes.

For 23. No, not transitive since $a \to b$ and $b \to c$, but not $a \to c$.

26. What are the equivalence classes of the equivalence relations in exercise 1.

In exercise 1, parts a and c were equivalence relations.

a. Since elements are only equivalent to themselves, the equivalence classes are the four singletons: $\{0\}, \{1\}, \{2\}$, and $\{3\}$.

c. Since 0 and 3 are each only equivalent to themselves, while 1 and 2 are equivalent to each other, there are 3 equivalence classes and they are $\{0\}, \{1, 2\}$, and $\{3\}$.

28. What are the equivalence classes of the equivalence relations in exercise 3.

In exercise 3, only parts a and d were equivalence relations.

a. $\{(f, g) \mid f(1) = g(1)\}$.

For each real number y, the set of functions whose value at 1 is y is an equivalence class.

d. $\{(f, g) \mid \exists C\ \forall x\ f(x) - g(x) = C\}$. Take any function f, and its equivalence class is $[f]$, the set of all functions of the form $f(x) + C$.

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