The Mean Value Theorem
Math 120 Calculus I
D Joyce, Fall 2013

The central theorem to much of differential calculus is the Mean Value Theorem, which we'll abbreviate MVT. It is the theoretical tool used to study the first and second derivatives.

There is a nice logical sequence of connections here. It starts with the *Extreme Value Theorem* (EVT) that we looked at earlier when we studied the concept of continuity. It says that any function that is continuous on a closed interval takes on a maximum and a minimum value.

A technical lemma. We begin our study with *a technical lemma* that allows us to relate the derivative of a function at a point to values of the function nearby. Specifically, if $f'(x_0)$ is positive, then for x nearby but smaller than x_0 the values $f(x)$ will be less than $f(x_0)$, but for x nearby but larger than x_0, the values of $f(x)$ will be larger than $f(x_0)$. This says something like f is an increasing function near x_0, but not quite. An analogous statement holds when $f'(x_0)$ is negative.

Proof. The proof of this lemma involves the definition of derivative and the definition of limits, but none of the proofs for the rest of the theorems here require that depth.

Suppose that $f'(x_0) = p$, some positive number. That means that

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = p.$$

So you can make $\dfrac{f(x) - f(x_0)}{x - x_0}$ arbitrarily close to p by taking x sufficiently close to x_0. In particular, as p is positive, you can make sure that $\dfrac{f(x) - f(x_0)}{x - x_0}$ is positive by taking x sufficently close to x_0 (within $\epsilon = p/2$ of p will do that). That quotient will be positive if and only if both $f(x) - f(x_0)$ and $x - x_0$ have the same sign. Thus $x > x_0$ if and only if $f(x) > f(x_0)$. That means if x is near x_0 but greater than x_0, then $f(x)$ is greater than $f(x_0)$. Likewise, if x is near x_0 but less than x_0, then $f(x)$ is less than $f(x_0)$. Q.E.D.

Zero derivatives at extreme values. It follows directly from this technical lemma that if a function f takes on an extreme value (either a max or a min) at point x_0 in an open interval, then the derivative of f cannot be positive or negative, and therefore, assuming that f does have a derivative there, the derivative must be 0.

Note that this conclusion can be made only on the interior of an interval. If the extreme value occurs at an endpoint of a closed interval, then the derivative doesn't have to be zero there.

Graphically, $f'(x) = 0$ means the graph of the function has a horizontal tangent at x.

Extreme values and critical points.   One of the most important things to know about a function is where it takes on extreme values—the maximum and the minimum value. We've just seen that if a differentiable function takes on an extreme value at an interior point of the function's domain, then the derivative is 0 there. It could also be that an extreme value occurs at an endpoint of the function's domain. One other place extreme values can be found is where the function is not differentiable.

Some textbooks use the term "critical point" or "critical value" to indicate one of these possible places where extreme values might occur—where f' is 0, where f' doesn't exist, and the endpoints of the function's domain.

Rolle's theorem.   Next, we'll consider a special case of the MVT, called *Rolle's theorem*. It's easier to prove Rolle's theorem because it's a special case, but the whole MVT can then, in turn, be proved from Rolle's theorem.

Rolle's theorem says if a function is continuous on a closed interval and differentiable on its interior, and it takes the same value at both endpoints of the interval, then somewhere on the open interval the function has a zero derivative.

Rolle's theorem follows from the EVT which guarantees that the function has a maximum and a minimum value, and the note above that says the derivative must be 0 at such an extremum if it occurs at an interior point. But if both extrema occur at the endpoints, then, since the function has the same value at both endpoints, the function is constant, so the derivative is 0 at every interior point.

It's interesting that Rolle was skeptical of the calculus but his theorem became central to the foundations of calculus. He was skeptical because at his time the foundations of calculus were very shaky. He developed his theorem as part of a different subject, namely the theory of equations, which is the study of finding solutions to polynomial equations. You'll see how it can be used for that purpose in some exercises.

The Mean Value Theorem.   The MVT says that if a function is continuous on a closed interval and differentiable on its interior, then somewhere on the interior of the interval its derivative equals the average rate of change of the function. That is, there is some point c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The MVT easily follows from Rolle's theorem by applying Rolle's theorem to the function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a} (x - a).$$

Just check first that $g(a) = g(b)$, then check that if $g'(c) = 0$, then $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ as required.

Theorems on increasing and decreasing functions.   The technical lemma that we looked at before really isn't what we want. It's too weak. But now that we have the MVT, we can show stronger statements.

Theorem. If the derivative of a function is positive on an interval, then the function is increasing on that interval; if negative, then decreasing; and if 0, then constant.

Proof. To prove this theorem, apply the MVT to pairs of points in the interval. Let $a < b$ on the inter val. By the MVT $\dfrac{f(b) - f(a)}{b - a}$ equals the value of the derivative at some point c between a and b.

First, suppose that f' is positive. Since $f'(c)$ is positive, therefore that quotient is positive. And the denominator $b - a$ is positive, so the numerator $f(b) - f(a)$ is also positive. Thus, $f(a) < f(b)$. Hence f is increasing.

The second part, f' negative implies f decreasing, is similar.

Finally, suppose that f' is 0 on the interval. Since $f'(c)$ is 0, therefore the quotient is 0. Therefore, the numerator $f(b) - f(a)$ is 0. But that says $f(a) = f(b)$, that is, f has the same value everywhere on the interval. $\qquad$ Q.E.D.

Note that that converse of this theorem almost holds. You can't quite conclude that if a differentiable function is increasing then its derivative is positive; it only follows that the derivative is nonnegative. For example, x^3 is an increasing function, but its derivative at 0 is not positive.

Functions with equal derivatives. We can also use the MVT to conclude that if two functions have equal derivatives on an interval, then they differ by a constant. For if $f' = g'$, then $(f - g)' = 0$, therefore, by the preceding theorem, $f - g$ is constant.

This theorem implies that if you know the derivative of a function, then you almost know the function.

This theorem will become important when we study integration. It says that two antiderivatives of a given funtion differ by a constant. That constant will be called the *constant of integration*, and it will be usually denoted C.

The first-derivative test. The first-derivative test tells you what kind of extremum you've got when the sign of the derivative changes. If f' changes from negative to positive at x, then f has a local minimum at x, but if f' changes from positive to negative at x, then f has a local maximum at x.

The argument that this test works comes from the theorems on increasing and decreasing functions. Consider the when f' changes from negative to positive. Then f changes from decreasing to increasing, so f has a minimum there.

The second-derivative test. In practice, the second-derivative test is easier to apply since you only have to know what's going on at one particular value of x. The test says that if f' is 0 at a point, and f'' is positive there, then f has a local minimum there. But if f' is 0 at a point, and f'' is negative there, then f has a local maximum there.

Let's look at the proof for the first case where $f''(x_0)$ is positive, and $f'(x_0) = 0$. Apply the technical lemma that started this discussion to f'. Since $f''(x_0)$ is positive, therefore for nearby x less than x_0, $f'(x) < f'(x_0)$, but for nearby x greater than x_0, $f'(x) > f'(x_0)$. But $f'(x_0)$ is 0, so f' changes from negative to positive at x_0. Therefore, by the first-derivative test, f has a minimum at x_0. The second case where $f''(x_0)$ is similar.

Warning: The second-derivative test is inconclusive when $f''(x_0) = 0$. If that happens use the first-derivative test. (Yes, you could look at the third derivative, but we won't go there.)

Concavity and the second derivative.   There are three equivalent conditions for a differentiable function to be concave up. The equivalence can be proved by means of the properties of the first and second derivatives described above. Any one of them could be used as the definition. A function is said to be *concave up* on an interval if any one of the following equivalent conditions holds:

(1). All the chords joining two points on the graph of the function lie above the graph of the function.

(2). All the tangents to the graph of the function lie below its graph.

Assuming that the function has a first derivative on the interval, another equivalent condition is

(3). The derivative of the function is increasing on the interval.

Assuming that the function has a second derivative on the interval, a fourth equivalent condition is

(4). The second derivative of the function is positive.

Similarly, we can define *concave down*.

Note than near a local maximum, a function is concave down, but near a local minimum, concave up.

Points of inflection.   If a function changes from concave up to concave down, or vice versa, at a point, then the function is said to have a *point of inflection* there. Note that the tangent line at a point of inflection crosses the graph of the function at that point. Also, if a function has a second derivative at a point of inflection, then the second derivative will have to be 0.

Quadratic functions have no points of inflection. Each one is either concave up everywhere or concave down everywhere. Every cubic function, however, will have exactly one point of inflection.

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