Changing Quantities
Math 120 Calculus I
D Joyce, Fall 2013

Unchanging quantities in algebra. You've used variables like x and y a lot in algebra and other courses before coming to calculus, but we're going to treat variables differently in this course.

When you first started using the symbol x in algebra, it was an unknown. There weren't different values of it, just one unknown value and it was your job to figure out what it was.

For example, here's a simple problem of that kind. Mary is three times as old as John. In five years, Mary will be twice as old as John is. How old is John now? Here's one solution: let x be how old John is now. Then Mary is $3x$ years old. In five years John will be $x+5$ years old while Mary will be $3x+5$ years old. Since Mary will be twice as old as John then, therefore $3x+5 = 2(x+5)$. Solving, $x = 5$.

Note how the variable didn't vary. The value of x didn't change at all.

In forumlas, like the volume V of a ball in terms of its ratius r, $V = \frac{4}{3}\pi r^3$, the variables V and r are fixed. One of them isn't known and your job is to find the other, but the variables don't vary.

Changing quantities in calculus. In calculus all our variables do vary. We consider them not with just one value but with many. Typically, the value of one variable depends on the value of another. When that's the case, and we don't have a good reason for choosing other letters for our variables, we'll let y be the variable that depends on the value of the variable x. Usually, then, x is our independent variable and y is our dependent variable.

When our variables have particular meaning, we'll use other letters like in the formula $V = \frac{4}{3}\pi r^3$. Frequently, our independent variable will be time, and we'll use t for that. Sometimes it will be something else like p for price.

Here are a couple of typical situations we'll consider, the first being where the independent variable is time.

Consider a ball tossed vertically up in the air. Let t be how long since it's tossed, and let y be how high it is in the air. As time passes, the ball goes up, and later comes down. Not only does value of y depend on the value of t, but we consider both to have lots of values. As time t passes, y first increases, then y decreases.

Next consider the revenue that you get at a retail store for some item depending on the price you set for that item. Let p be the price, and let R be how much money you get for selling that item. Depending on the price p you set, you'll get more or less revenue R. If p is too large, no one will buy it, and R will be 0. If you set p to 0, people may take it away from your store, but again R will be 0. In between, R will be positive, but it depends on p. Time doesn't enter into this example. Instead, p is the independent variable. Like in the last example, R and p take lots of values, but R depends on p.

The variables we take for the independent and dependent variables depend on the application.

Functional notation. $y = f(x)$.

For the longest time, from the 1500s when symbolic algebra was invented through the 1700s, variables were enough. If y depended on x, you could say that in words or express it as a formula, like $y = x \sin x$.

The notation itself is a useful one. If you have $y = f(x) = x \sin x$, then you can us the operation of substitution to evaluate y for different values of x. For example, when $x = \frac{\pi}{2}$, then $y = f(\frac{\pi}{2}) = \frac{\pi}{2} \sin \frac{\pi}{2} = \frac{\pi}{2}$. Besides substituting values for x, you can also substitute expressions. So, for instance, $f(u+v) = (u+v)\sin(u+v) = (u+v)(\sin u \cos v + \sin v \cos u)$. We'll be doing that a lot in this course.

Mathematicians started using functional notation at the end of the 1700s. Giving a name, f to the dependency of y on x allows us to thing of that dependency as a thing in itself. It makes a noun, the function f, out of a verb "depends", and it allows us to do things that we couldn't easily do otherwise.

For example, we'll compute derivatives of functions. Later in the course, we'll be able to compute the derivative of the function $f(x) = x \sin x$ and find that it's $f'(x) = \sin x + x \cos x$.

Uniform change. This is something you know about, and you don't need calculus for it. When the rate of change of the dependent variable y is always that same as the independent variable x changes, you've got uniform change. For uniform change, the change in y is proportional to the change in x.

For example, if you're walking down the street at a constant velocity of 3 miles/hour, then the distance travelled is proportional to the elapsed time, and the constant of proportionality is 3 miles/hour. In 1 hour, you go 3 miles. In half an hour you go $\frac{1}{2}$ times 3, or 1.5 miles. And so forth. For uniform motion, distance = rate times time.

The ancient Greek mathematicians understood uniform change well.

Autolycus of Pitane (ca. 360 BCE–ca. 290 BCE) studied the circular uniform motion of stars in the heavens. He said an object moved uniformly if it traversed equal distances in equal times. That means that for each fixed time period, the object moves the same distance.

Archimedes (ca. 287 BCE–ca. 212 BCE) used this definition to prove that an object moving uniformly traverses distances proportional to times. That means that the distance x traversed in a time interval of length t is proportional to the distance y traversed in a time interval of length s:

$$\frac{x}{y} = \frac{t}{s}.$$

Note that in this statement there is no mention of velocities, only of distances and times. The ancient Greeks accepted ratios of two quantities as long as they were of the same kind. Here we have the ratio of two distances equal to the ratio of two time intervals. But the ancient Greeks did not accept mixed ratios, for instance the ratio of a distance to a time.

Once mixed ratios were accepted, the above proportion could be written in the alternate form

$$\frac{x}{t} = \frac{y}{s}.$$

We interpret that as saying the velocity x/t during the first time interval equals the velocity y/s during the second time interval. In other words, an object moves uniformly when it has a constant velocity. The modern formula for uniform motion says that the distance traveled equals the product of the rate (or velocity) and the elapsed time.

Unfortunately, they didn't understand nonuniform motion. In order to deal with the motion of the planets, they made all motions compounded out of uniform circular motions. And although that made their astronomy complicated, it was still computable. The Ptolemaic planetary system was made of dozens of circles with planets revolving around the earth each compounded of several uniform circular motions. Even when Copernicus used the sun as the center of his planetary system in the early 1500s he still used compounded circular motions.

It wasn't until Kepler used nonuniform motion in the early 1600s that compounded uniform circular motion became unnecessary.

Nonuniform change. This is what calculus deals with. For nonuniform motion, the rate of change isn't constant.

If an object moves fast sometimes and slow other times, that's nonuniform motion. A ball tossed in the air is nonuniform.

Sales revenue depending on price is non uniform, too, since when you first increase the price starting at 0, the revenue increases fast, then increases slower, then decreases until eventually the revenue reaches 0. The rate of change of revenue with respect to price is not constant. Another way of saying that is that the change in revenue is not proportional to the change in price. Sometimes it's positive, sometimes it's negative.

The ancient Greek mathematicians did not study nonuniform change. But by the 1300s scholars were comfortable with velocity, the rate of change of a changing quantity. They used the term velocity in a more general way than we do now. We use it as a the rate of change of an object that moves over time. For them, the dependent variable didn't have to be distance and the independent variable didn't have to be time; nonetheless, for purposes of exposition, let's limit ourselves to an object that moves over time.

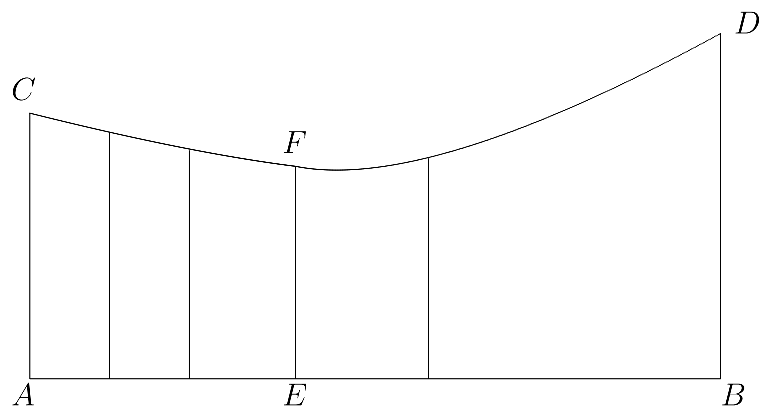
They were trying to understand nonuniform motion, that is, when the velocity is not constant. The problem was that velocity is only defined under uniform motion. What is velocity if the motion is not uniform?

Four of these scholars at Merton College in Oxford University—Thomas Bradwardine, William Heytesbury, Richard Swineshead, and John Dumbleton—studied nonuniform motion in the first half of the 1300s. Even though they couldn't precisely define velocity, they worked with velocities as if they were real quantities. Furthermore, they understood when the velocity was changing, it had a rate of change, the acceleration. (Or deceleration if the velocity was decreasing. They didn't know about negative numbers.)

The Merton mean speed theorem. One of their discoveries about a certain nonuniform motion is called the Merton Mean Speed Theorem. It says that if an object is moving with a constant acceleration, then the distance it travels is the same distance it would travel if it were moving at a constant velocity, that velocity being the average of its initial and final velocity. This happens to be the motion of a body in free fall, but there's no indication that these Merton scholars knew that. That result was something Galileo (1564–1642) discovered much later.

We'll look at one of the Merton scholar's proofs of this mean speed theorem in class.

Oresme's Fundamental Theorem of Calculus Nicole Oresme (ca. 1323–1382) was at the University of Paris and expanded the analytic study of changing quantities. He had a graphical interpretation very similar to the modern graph $y = f(x)$ of a function in the (x, y)-plane, although analytic geometry and coordinates were yet to be developed by Fermat and Descartes in the 1600s.



He represented time as a line, much as Aristotle had done long before, so that instants in a time interval were represented by points on a horizontal line segment AB, which he called the longitude. Given a moving object, at each instant in time E that moving object has a velocity, and he represented that velocity by a vertical line segment EF proportional to the velocity; each vertical line segment he called a latitude. These latitudes together formed a plane region $ABDC$, which he called a form, bounded on the bottom by the original longitude AB, on the left by the initial latitude AC representing the initial velocity, on the right by the final latitude BD representing the final velocity, and on the top by the curve CFD which he called the summit curve. He then argued that the area of that form $ABDC$ is proportional to the distance traveled.

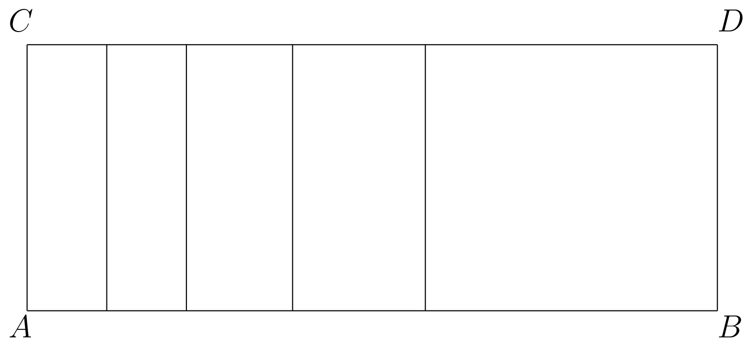
Oresme made the various lengths proportional to distances or times since he thought (as the ancient Greeks did) that they're different kinds of things, but we would probably use the language of equality: we would make length of the longitude equal the elapsed time, and the length of a latitude equal the velocity at that instant.

We can write his result using Leibniz' notation as

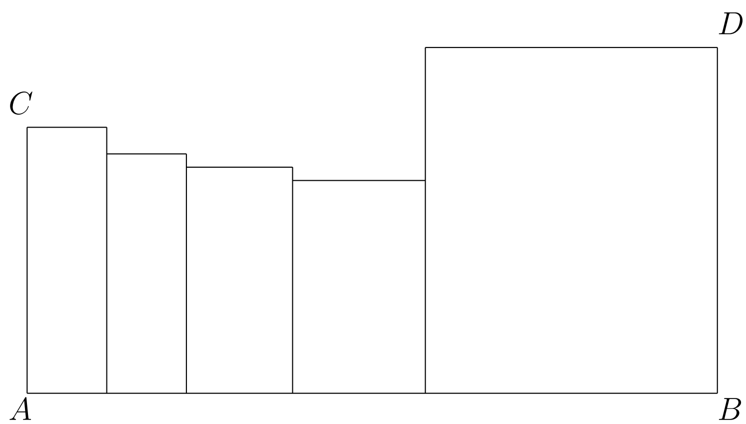
$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

In the 1300s there was no symbolic algebra at all—no equal sign, no minus sign, and no variables.

Oresme gave examples of this principle and used it to prove the Merton mean speed theorem.



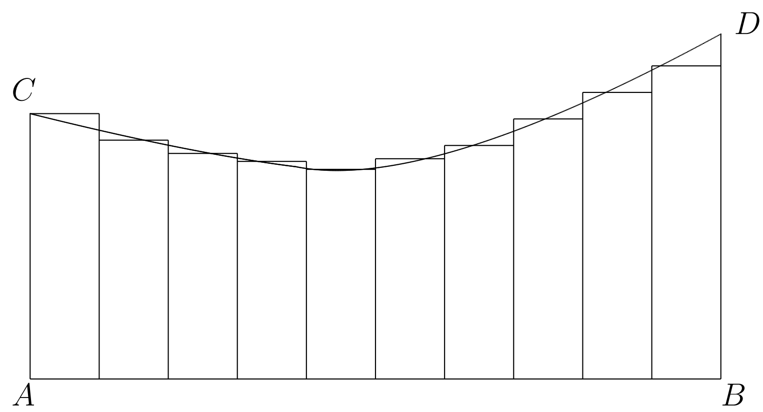
Here's the form for uniform motion. Since the velocity is constant for uniform motion, all the latitudes are equal, so the form has a horizontal summit, that is, the form is a rectangle. The distance traveled is that velocity times the elapsed time, but the area of the rectangle is also the velocity (length of the latitudes) times the elapsed time (length of the longitude).



Next, consider an object moves in a series of uniform motions. Over the first time interval, it has one velocity, over the second time interval a second velocity, and so forth. The form is made of a first rectangle, a second rectangle just to its right, and so forth. The area of the form is the sum of the areas of the rectangles, but the area of each rectangle is the distance traveled over the corresponding time interval, so the area of the form is the total distance traveled.

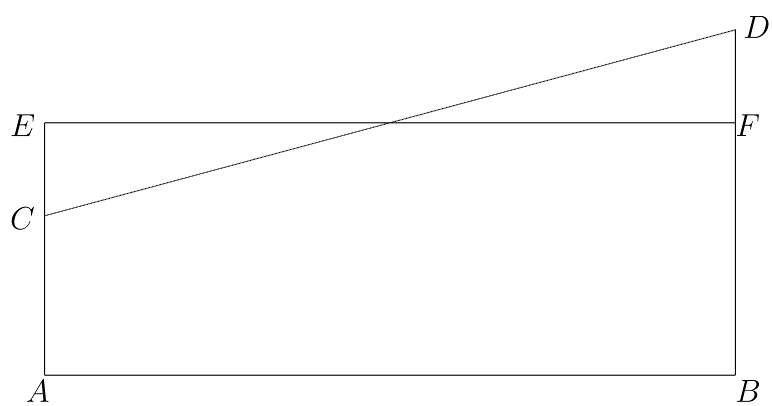
Finally, suppose that the object has a nonuniform motion. The velocity is not constant anywhere, but changes. The form has a curve at the top. There are a couple of ways to be convinced that this principle (FTC) applies in this case also.

One argument is to employ some kind of concept of limiting approximations. Here's one argument of that type. Divide the latitude into small subintervals and assume that the velocity during each subinterval is constant, say the actual velocity at some instant in that subinterval. That gives the motion as described in the previous case, so the total area of the rectangles is the distance traveled for that motion. Since the velocity for that motion is close to the velocity of the original nonuniform motion, therefore the area of the curved form is approximately the distance traveled for that nonuniform motion. Some kind of limiting argument is needed to complete the argument. (Methods of this sort were used later by Fermat, Newton, Riemann, and Darboux.)



An alternate argument for FTC relies on a different interpretation of motion. Aristotle had said that motion takes place over an interval. So velocity is defined over an interval of time rather than at a point in time. One interpretation of that is that even for nonuniform motion, velocity is constant over intervals, although they are very short intervals. Then the motion actual is a series of uniform motions and the previous case already does the general case. (This is very similar to Leibniz' approach to motion where the subintervals of constant velocity are infinitesimals.)

Oresme's proof of the Merton mean speed theorem. Oresme used this principle to prove the Merton mean speed theorem. Suppose that an object undergoes constant acceleration. Then its velocity increases (or decreases) at a constant rate. Therefore the summit line of the form is a slanted straight line. (The complete argument that it's a straight line involves similar triangles.) Thus, the form is a trapezoid $ABDC$ with vertical parallel sides AC and BD.



But the area of this trapezoid $ABDC$ is the same as the area of a rectangle $ABFE$ on the same base but whose height $AE = BF$ is the average height of the left and right sides of the trapezoid. Thus the distance traveled under constant acceleration is the same as the distance traveled by an object going a constant velocity, that being the average of the initial and final velocities.

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