Math 128, first test sample problems.

These are sample questions. You may bring one sheet of prepared notes for the test. Calculators are optional; you may bring one if you like.

Problem 1. On complex numbers.
   a. Write the reciprocal of $5 + 12i$ in the form $a + bi$.
   b. What are the two square roots of $i$?
   c. Write $e^{i\pi/3}$ in the form $a + bi$.
   d. Write $5 + 5i$ in polar form $re^{i\theta}$.

Problem 2. On transformations.
   a. Let $S$ and $T$ be two translations on the complex plane $\mathbb{C}$. Prove that their composition $T \circ S$ is also a translation.
   b. The transformation $T(z) = iz + (1 - i)$ is a rotation. Determine (1) its fixed point, and (2) its angle of rotation.
   c. Give a formula for the transformation $T$ of the complex plane $\mathbb{C}$ which is a scaling (that is, a homothetic transformation) that fixes $0$ and sends $3i$ to $4i$.

Problem 3. On the stereographic projection. Recall that with the stereographic projection the point $(a, b, c)$ on the unit sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ corresponds to the point $z = \frac{a + ib}{1 - c}$ in $\mathbb{C}^+$. The only exception to this correspondence is that the “North pole” $(0, 0, 1)$ of the sphere $S^2$ corresponds to the point $\infty$ in $\mathbb{C}^+$.

The plane $x + y + z = 1$ in space $\mathbb{R}^3$ intersects the unit sphere $S^2$ in a circle. By means of the stereographic projection, that circle corresponds to some curve in $\mathbb{C}^+$. Describe either in words or by means of an equation what curve that is. Explain your answer.

Problem 4. On transformation groups. Consider the collection $G$ of transformations of the plane $\mathbb{C}$ that includes all translations and all half turns.
   a. In order for $G$ to be a transformation group, three conditions have to be satisfied. State each condition, and after you state it, explain why $G$ satisfies that condition.
   b. Let $ABC$ be an equilateral triangle. Describe an equilateral triangle $DEF$ whose sides are equal to those of $ABC$, but, with respect to the group $G$, triangle $DEF$ is not congruent to triangle $ABC$.

Problem 5. On Möbius transformations.
   a. There is one Möbius transformation $T$ that maps $0 \mapsto 1$, $\infty \mapsto 0$, and $1 \mapsto \infty$. What is it?
   b. Determine the fixed points of the transformation $T$ you found in part a.

Problem 6. Prove that the only Möbius transformations that have a single fixed point at $\infty$ are translations.