Math 128, Modern Geometry

D. Joyce, Clark University

25 Aug 2005

There's more than one kind of geometry

You have probably only seen one kind of plane geometry so far, classical Euclidean geometry, but you may have seen it from two different points of view. One is usually called "synthetic geometry" and is what Euclid did and what is presented in a typical plane geometry course in high school. It starts with axioms about points, lines, angles, and triangles and derives theorems from the axioms such as the Pythagorean theorem. The other point of view is called "analytic geometry" and studies geometry with the help of algebra and equations. The plane is coordinatized, so each point has an "x-coordinate" and a "y-coordinate," then various curves are described by equations—straight lines by linear equations $ax + by = c$; circles, ellipses, parabolas, and hyperbolas by various quadratic equations. Using analytic geometry, questions in geometry can be translated into questions in algebra.

We will review part of Euclid's *Elements*, but it will be more than a review because it will give a point of departure. As we go over it, I will introduce some other kinds of geometry, in particular, elliptic and hyperbolic geometry that arise by denying the parallel postulate. The parallel postulate of Euclidean geometry is the one which implies that given a line and a point not on the line there is exactly one line through the given point parallel to the given line. If you postulate that there is more than one, then you get hyperbolic geometry, but if you postulate there are none at all, then you get elliptic geometry. These were developed in the nineteenth century when it was finally realized that the parallel postulate did not logically follow from the other axioms of Euclidean geometry. They have since found important applications in modern mathematics. We'll study those geometries in more detail, as wells as other geometries.

Other kinds of geometry arise when you forget certain aspects of classical Euclidean geometry. For instance, when you only remember about points, straight lines, and incidence (when a point lies on a line), then you are studying affine geometry. Affine geometry does not consider distance or angles. You might say affine geometry is a fragment of Euclidean geometry. Most of the theorems of affine geometry are vastly simplified by adding "points at infinity" so that parallel lines meet "at infinity." Then you get projective geometry. Projective geometry also doesnt consider distance or angles, and it may be considered to be a fragment of elliptic geometry.

If you remember a different part of classical Euclidean geometry instead, namely "betweenness" (when one point lies between two other points on a line), you get ordered geometry. Because Euclid never explicitly mentioned order, many of the proofs of his theorems are logically flawed. This oversight was not corrected until the end of the nineteenth century and the beginning of the twentieth. Since then, its been recognized that the correct way to do synthetic geometry is to begin with axioms of betweenness as well as the axioms Euclid used (or others of equivalent strength, of course).

What is geometry?

If there's more than one kind of geometry, then that brings up the question: what is geometry? What counts as geometry?