Math 128, Modern Geometry

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Due Today. From Chapter 8: 1, 2, 8.

Last time. We finished our study of the hyperbolic plane. We looked at area. We saw that the area of a triangle was equal to the angle defect of the triangle, and that implied that that the largest triangle has area π. Circles, however, have area that grows exponentially with the radius.

Today. We'll look at elliptic geometry. There are two kinds of elliptic geometry, one called double elliptic geometry, the other called single elliptic geometry. We'll get double elliptic geometry using the Erlanger programm from Möbius geometry. That is, we'll specify a particular subgroup of the Möbius group, and its geometry we'll call double elliptic geometry.

By means of the stereographic projection, we'll see double elliptic geometry as the geometry of the sphere where its straight lines are great circles of the sphere. Since any two great circles intersect at two antipodal points, that means that two lines in double elliptic geometry will intersect at two points rather than one. Either by identifying each pair of antipodal points to a single point, or by selecting one of the pair, we'll get single elliptic geometry. In single elliptic geometry any two straight lines will intersect at exactly one point. But the single elliptic plane is unusual in that it is unoriented, like the Möbius band.

The elliptic group and double elliptic geometry. The group of transformation that defines elliptic geometry includes all those Möbius transformations T that preserve antipodal points. On the sphere that means if P and P^* are antipodal points, then $T(P)$ and $T(P^*)$ are also antipodal points. By the stereographic projection, antipodal points correspond to points z and z^* of $\mathbf{C}^+$ such that $\overline{z}z^* = -1$. With a bit of algebra, as shown in the text, such a transformation can always be written in either of these two forms:

$$T(z) = \frac{az+b}{-\overline{b}z+\overline{a}} \text{ where } |a|^2 + |b|^2 = 1$$

or

$$T(z) = e^{i\theta}\frac{z-z_0}{\overline{z}_0 z+1}.$$

This group defines the *double plane elliptic geometry* on $\mathbf{C}^+$. Since the transformation preserve antipodal points, great circles are sent to great circles, and these are the straight lines in elliptic geometry. Actually, great circles correspond to certain circles in $\mathbf{C}^+$, namely those that meet the unit circle at ends of diameters.

Single elliptic geometry. As mentioned above, lines in double elliptic geometry meet at two points instead of one. To define single elliptic geometry, use the same group, but identify antipodal points, that is, if $\overline{z}z^* = -1$, then z and z^* in $\mathbf{C}^+$ refer to the same point in the single elliptic plane.

An alternate model for the single elliptic plane is to take the interior of the unit disk along with half the unit circle. That way one of each pair of antipodal points is selected to be the point of the single elliptic plane. When you do this, however, you have to remember that points on one side of the unit disk are close to points on the other side.

Most of the axioms for Euclidean geometry also hold in single elliptic geometry. The parallel postulate does not, of course, since all lines intersect in elliptic geometry, so there are not parallel lines there. A couple of the others require interpretation to make them work in elliptic geometry. The first 15 propositions in Book I of Euclid's *Elements* do hold in elliptic geometry, but not the 16th. We'll look at it.

Length and area. Length and area are defined so that they can be measured in the usual way on the sphere via stereographic projection. That's the unit sphere, so the length of any of its great circles is 2π. Since half of a great circle corresponds to a line in the single elliptic plane, the total length of a such a line is π. Lines have finite length in elliptic geometry.

Triangles in the elliptic plane have an angle sum greater than 180°. The *angular excess of a triangle* is defined by how much greater than π the angle sum of the triangle is. That is, if α, β, and γ are the three angles of the triangle, then its angle sum is $\alpha+\beta+\gamma-\pi$. Just as the angle defect is additive in hyperbolic geometry, the angle excess is additive in elliptic geometry. Therefore, the area of a triangle in the elliptic plane is proportional to the angle excess. We've already scaled everything properly so the constant of proportion is 1, and therefore, the area of a triangle equals the angle excess of the triangle.