Math 128, Modern Geometry

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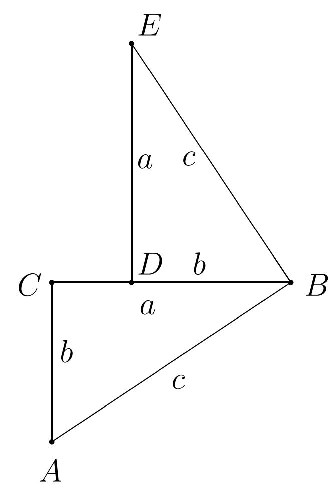
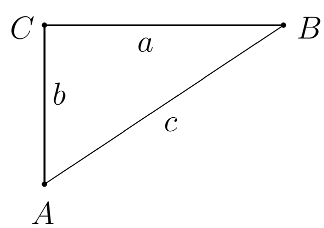
Last week we looked at the assumptions of the proof the the Pythagorean theorem in more detail. Here's a modified proof with some of those details included.

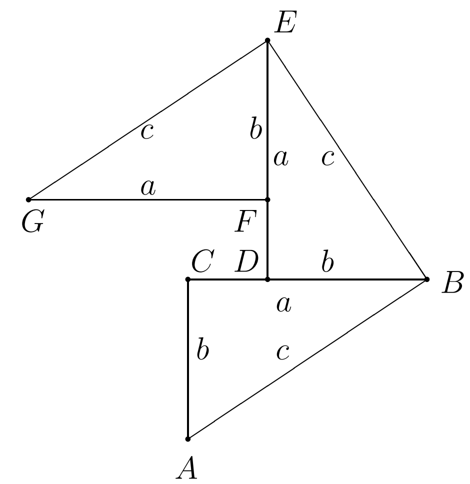
Given information. Let ABC be a right triangle with right angle at C, and let a, b, and c be the sides opposite the vertices A, B, and C, respectively.



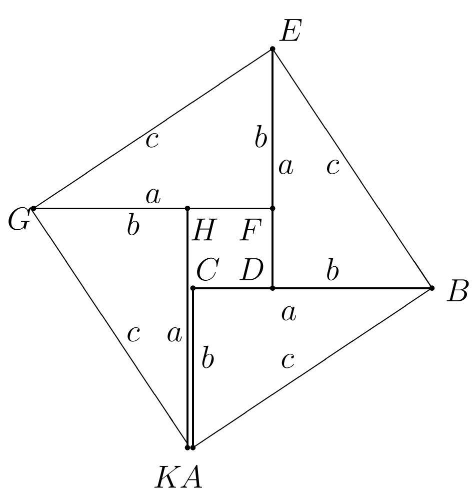


Part I. The construction of the figure. Apply an isometry of the plane which moves A to B and moves C to a point D on the line CB. Let that isometry move the point B to the point E on the other side of the line CB. This results in a triangle BDE congruent to the original triangle ACB. Therefore, $DE = a$, $BD = b$, and $BE = c$.

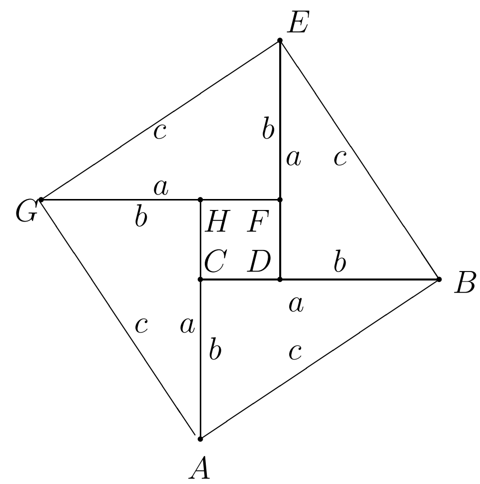
Next, apply an isometry of the plane which moves B to E and moves D to a point F on the line DE. Let that isometry move the point E to a point G on the other side of the line DE. This results in a triangle EFG congruent to triangle BDE, and, therefore, congruent to the original triangle ACB. Therefore, $FG = a$, $EF = b$, and $EG = c$.



Next, apply an isometry of the plane which moves E to G and moves F to a point H on the line FG. Let that isometry move the point G to a point K on the other side of the line FG. This results in a triangle GHK congruent to triangle EFG, and, therefore, congruent to the original triangle ACB. Therefore, $FG = a$, $EF = b$, and $EG = c$.



Then K coincides with A (we have to figure out why), and C lies on the line AH (again, we have to figure out why). That gives us a figure with four congruent triangles.



Part II. Why the inner and outer figures are squares. We define a *square* as a four-sided figure whose four sides are all equal and whose four angles are all right. Since the four triangles are congruent, we know the four sides of the outer figure all have the same length c. The four sides of the inner figure are each differences between line segments of length a and line segments of length b. Assuming $b > a$, that implies each side of the inner figure has length $b - a$. (If $a < b$ or $a = b$, there are other cases to consider.)

We still have to show the the four angles of the outer and inner figures are all right angles. A *right angle* is defined as being an angle made when a ray meets a line making two equal angles. By assumption, the original triangle ABC has a right angle at C. That means angle ACB and its supplement, angle HCB are both right angles. Since the corresponding angles of the other three congruent triangles are equal, that means we have eight right angles in the diagram, namely, ACB, HCB, BD, CDE, EFG, DFG, GHA, and FHA. Four of these are the angles of the inner figure; therefore, the figure $CDFH$ is a square.

For the outer figure to be a square, we have to show its four angles, GAB, ABE, BEG, and EGA are right angles. Each is the sum of two angles of the four congruent triangles, and, since they are congruent triangles, each is the sum of equal angles, so all four angles of the outer figure are equal. But we have yet to show they are right angles. Take one of them, angle GAB. It is the sum of the two angles GAH and CAB. But angle GAH equals angle ABC, so the outer angle GAB is the sum of two angles ABC and CAB of the right triangle ABC, where the third angle ACB is a right angle. But the sum of the three interior angles of any triangle equals 2 right angles (we recognize that as a proposition that this proof relies on), therefore, angle GAB equals one right angle. Likewise the other three angles of the outer figure are right angles. Therefore, figure $ABEG$ is a square.

Part III. The derivation of the Pythagorean identity. Then the large $c \times c$ square is made out of four congruent triangles each with area $ab/2$ and the square hole of area $(b-a)^2$. Therefore,

$$c^2 = 4\,\frac{ab}{2} + (b-a)^2 = 2ab + b^2 - 2ab + a^2 = a^2 + b^2.$$

Q.E.D.

That's pretty much the way things stand now. We've identified some definitions and a major proposition (about the interior angle sum of a triangle), as well as a few minor points, and a major gap (why K coincides with A and the lines AC and KH are the same) in the proof.