Math 128, Modern Geometry

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We've analyzed a nice proof of the Pythagorean theorem and found that it depends on a number of assumptions. Some of these assumptions are merely definitions, and some we found we could prove based on what we consider more primitive assumptions.

Although we haven't finished the process of analysis, it's time we wrote up what we've got. When we write it up, we won't do it in the order that we discovered things; instead, we'll write it up in a logical order so that assumptions that are used in proofs of theorems appear before the theorems appear. That means the Pythagorean theorem will appear last, those theorems it depends on in the middle, and the statements those depend on at the beginning.

Definitions and unproved statements.

Definition. When one straight line meets another straight line making two equal angles, those two equal angles are called *right angles*, and the two straight lines are said to be *orthogonal* or *perpendicular* lines.

Definition. A *right triangle* is a triangle one of whose angles is a right angle. The side opposite the right angle is called the *hypotenuse* of the right triangle, and the other two sides its *legs*.

Definition. Two triangles, ABC and DEF, are *congruent* if corresponding angles and sides are equal, that is, angle A equals angle D, angle B equals angle E, angle C equals angle F, side AB equals side DE, side BC equals side EF, side CA equals side FD.

Definition. A *parallelogram* is a four-sided figure having parallel opposite sides.

Definition. A *rectangle* is a four-sided figure having four right angles.

Definition. A *square* is a rectangle having four right angles and four equal sides.

Unproved statement. (SAS congruence theorem.) If two triangles, ABC and DEF, have angle A equal to angle D, angle B equal to angle E, and included side AB equal to included side DR, then the two triangles are congruent.

Unproved statement. The alternate interior angles a transversal line makes with two parallel lines are equal

Unproved statement. The sum of the three interior angles of any triangle equals two right angles.

Unproved statement. Two lines that are perpendicular to the same line are parallel.

Unproved statement. Lines parallel to perpendicular lines are perpendicular to each other.

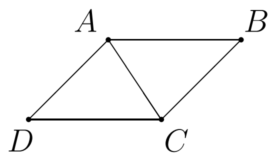
Unproved statement. The area of a right triangle is half the product of its legs.

Unproved statement. The area of a square is the product of two of its sides.

Intermediate statements that we've proved, i.e., theorems.

Theorem. Opposite sides of a parallelogram are equal.

Proof: Let $ABCD$ be a parallelogram. Draw a diagonal line AC connecting opposite vertices of the parallelogram.



The alternate interior angles this transversal AC makes with the parallel lines AB and CD are equal, so angle BAC equals angle DCA. Likewise, he alternate interior angles it makes with the parallel lines AD and BC are equal, so angle DAC equals angle BCA. We have two triangles, namely, triangle BAC and triangle DCA, with two equal corresponding angles, and the side between those angles, namely, AC is equal in both triangles; therefore, these two triangles are congruent. Therefore, the corresponding sides AB and CD of these two congruent triangles are equal. Thus, we have shown that the opposite sides of the parallelgram are equal. Q.E.D.

Theorem. Opposite sides of a rectangle are equal.

Proof: Since opposite sides of a rectangle are perpendicular to the remaining sides, therefore opposite sides are parallel. Thus, a rectangle is a parallelogram. But parallelograms have equal opposite sides, so rectangles do, too. Q.E.D.

The Pythagorean theorem. The square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides.

Proof: Let ABC be a right triangle with right angle at C, and let a, b, and c be the sides opposite the vertices A, B, and C, respectively.

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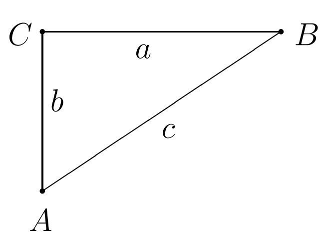
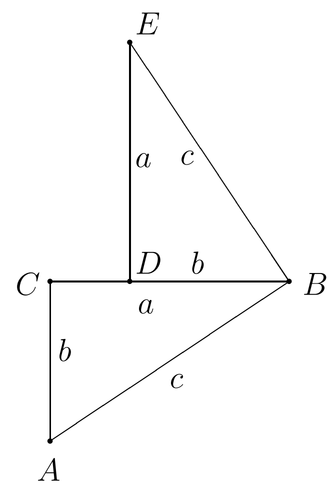
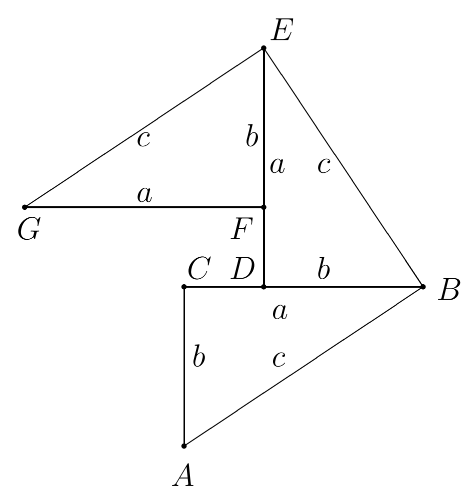


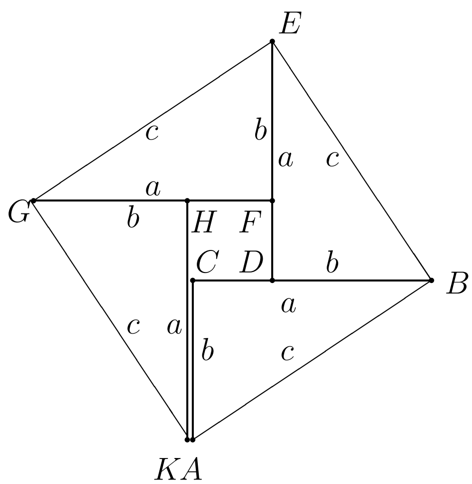
Part I. The construction of the figure. Apply an isometry of the plane which moves A to B and moves C to a point D on the line CB. Let that isometry move the point B to the point E on the other side of the line CB. This results in a triangle BDE congruent to the original triangle ACB. Therefore, $DE = a$, $BD = b$, and $BE = c$.



Next, apply an isometry of the plane which moves B to E and moves D to a point F on the line DE. Let that isometry move the point E to a point G on the other side of the line DE. This results in a triangle EFG congruent to triangle BDE, and, therefore, congruent to the original triangle ACB. Therefore, $FG = a$, $EF = b$, and $EG = c$.



Next, apply an isometry of the plane which moves E to G and moves F to a point H on the line FG. Let that isometry move the point G to a point K on the other side of the line FG. This results in a triangle GHK congruent to triangle EFG, and, therefore, congruent to the original triangle ACB. Therefore, $FG = a$, $EF = b$, and $EG = c$.

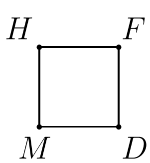


Now, two lines that are perpendicular to the same line are parallel. In our figure, HF and CD are both perpendicular to DF, so they're parallel. Also, LH and DF are perpendicular to HF, so they're parallel.

Next, lines parallel to perpendicular lines are perpendicular to each other. Since $HF \| CD$, and $LH \| DF$, and $HF \perp DF$, therefore $HL \perp CD$.

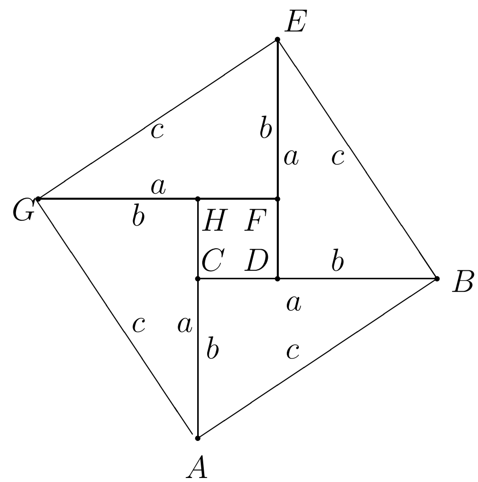
Therefore, lines HL and DC meet at some point M with a right angle at M.

In the central figure



all four angles are right, so $HFDM$ is a rectangle. Also, two adjacent sides, HF and FD, of this rectangle are equal, so the opposite sides MD and HM are also equal to them. Hence, $HFDM$ is a square.

Then C coincides with M since $CD = MD$ and they lie on the same line. Furthermore, this lines HC and HK coincide because angles FHC and FHK are both right angles. That gives us a figure with four congruent triangles surrounding a square in the middle.



Since the four triangles are congruent, we know the four sides of the outer figure all have the same length c.

For the outer figure to be a square, we have to show its four angles, GAB, ABE, BEG, and EGA are right angles. Each is the sum of two angles of the four congruent triangles, and, since they are congruent triangles, each is the sum of equal angles, so all four angles of the outer figure are equal. But we have yet to show they are right angles. Take one of them, angle GAB. It is the sum of the two angles GAH and CAB. But angle GAH equals angle ABC, so the outer angle GAB is the sum of two angles ABC and CAB of the right triangle ABC, where the third angle ACB is a right angle. But the sum of the three interior angles of any triangle equals two right angles, therefore, angle GAB equals one right angle. Likewise the other three angles of the outer figure are right angles. Therefore, figure $ABEG$ is a square.

Part II. The derivation of the Pythagorean identity. Then the large $c \times c$ square is made out of four congruent triangles each with area $ab/2$ and the square hole of area $(b-a)^2$. Therefore,

$$c^2 = 4\,\frac{ab}{2} + (b-a)^2 = 2ab + b^2 - 2ab + a^2 = a^2 + b^2.$$

Q.E.D.