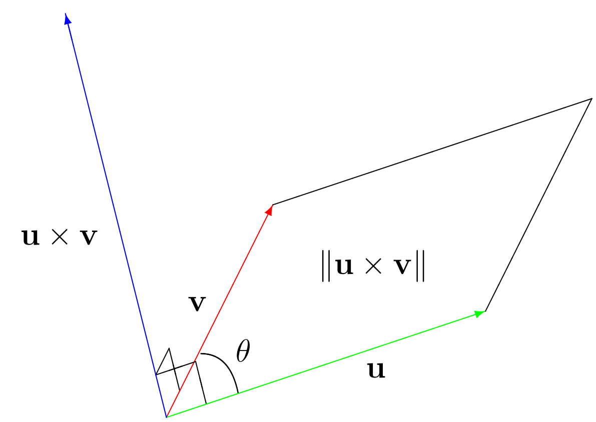
Cross products
Math 130 Linear Algebra

D Joyce, Fall 2015

The definition of cross products. The cross product $\times : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ is an operation that takes two vectors $\mathbf{u}$ and $\mathbf{v}$ in space and determines another vector $\mathbf{u} \times \mathbf{v}$ in space. (Cross products are sometimes called outer products, sometimes called vector products.) Although we'll define $\mathbf{u} \times \mathbf{v}$ algebraically, its geometric meaning is more understandable.



The cross product $\mathbf{u} \times \mathbf{v}$ is determined by its length and its direction. It's length is equal to the area of the parallelgram whose sides are $\mathbf{u}$ and $\mathbf{v}$, and that area is the length of $\mathbf{u}$ times the length of $\mathbf{v}$ times the sine of the angle θ between them. Thus

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos \theta$$

The direction of $\mathbf{u} \times \mathbf{v}$ will be orthogonal to the plane of $\mathbf{u}$ and $\mathbf{v}$ in a direction determined by a right-hand rule (when the coordinate system is right-handed).

The easiest way to define cross products is to use the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ for $\mathbf{R}^3$. If

$$\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k},$$

and

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k},$$

then $\mathbf{u} \times \mathbf{v}$ is defined as

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

which is much easier to remember when you write it as a determinant

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}
\end{aligned}$$

Properties of cross products. There are a whole lot of properties that follow from this definition. First of all, it's anticommutative

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}),$$

so any vector cross itself is $\mathbf{0}$

$$\mathbf{v} \times \mathbf{v} = \mathbf{0}.$$

It's bilinear, that is, linear in each argument, so it distributes over addition and subtraction, $\mathbf{0}$ acts as zero should, and you can pass scalars in and out of arguments

$$\mathbf{u} \times (\mathbf{v} \pm \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \pm (\mathbf{u} \times \mathbf{w})$$

$$(\mathbf{u} \pm \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) \pm (\mathbf{v} \times \mathbf{w})$$

$$\mathbf{0} \times \mathbf{v} = \mathbf{0} = \mathbf{v} \times \mathbf{0}$$

$$c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$$

A couple more properties you can check from the definition, or from the properties already found are that $\langle \mathbf{u}\times\mathbf{v} | \mathbf{u} \rangle = 0$ and $\langle \mathbf{u}\times\mathbf{v} | \mathbf{v} \rangle = 0$. Those imply that the vector $\mathbf{u}\times\mathbf{v}$ is orthogonal to both vectors $\mathbf{u}$ and $\mathbf{v}$, and so it is orthogonal to the plane of $\mathbf{u}$ and $\mathbf{v}$.

Standard unit vectors and cross products. Interesting things happen when we look specifically at the cross products of standard unit vectors. Of course

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0},$$

since any vector cross itself is $\mathbf{0}$. But

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j},$$

and

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j},$$

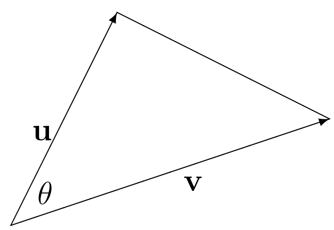
all of which follows directly from the definition.

Length of the cross product, areas of triangles and parallelograms. A direct computation (which we'll omit) shows that

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\sin\theta$$

where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

Consider a triangle in 3-space where two of the sides are $\mathbf{u}$ and $\mathbf{v}$.

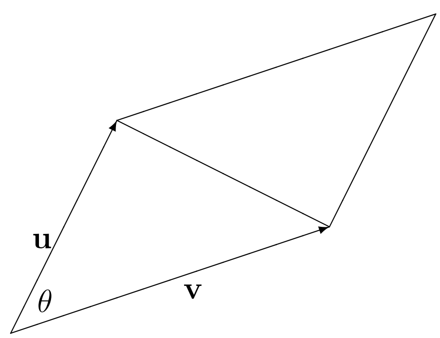


Taking $\mathbf{u}$ to be the base of the triangle, then the height of the triangle is $\|\mathbf{v}\|\sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Therefore, the area of this triangle is

$$\text{Area} = \tfrac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta = \tfrac{1}{2}\|\mathbf{u} \times \mathbf{v}\|.$$

(In general, the area of a any triangle is half the product of two adjacent sides and the sine of the angle between them.)

Area of a parallelogram in $\mathbf{R}^3$. Now consider a parallelogram in 3-space where two of the sides are $\mathbf{u}$ and $\mathbf{v}$.



Of course, if the triangle is doubled to a parallelogram, then the area of the parallelogram is $\|\mathbf{u} \times \mathbf{v}\|$.

Thus, the norm of a cross product is the area of the parallelgram bounded by the vectors.

We now have a geometric characterization of the cross product. The cross product $\mathbf{u} \times \mathbf{v}$ is the vector orthogonal to the plane of $\mathbf{u}$ and $\mathbf{v}$ pointing away from it in a the direction determined by a right-hand rule, and its length equals the area of the parallelgram whose sides are $\mathbf{u}$ and $\mathbf{v}$.

Note that $\mathbf{u} \times \mathbf{v}$ is $\mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ lie in a line, that is, they point in the same direction or the directly opposite directions.

Math 130 Home Page at `http://math.clarku.edu/~ma130/`