We’d like to be able to say that the dimension of a vector space is the number of vectors in its basis, but we can’t yet since we haven’t yet proved that any two bases of a vector space have the same number of vectors. We’ll do that, at least in the case that a vector space has a finite basis. If a vector space doesn’t have a finite basis, it will have an infinite dimension. We’ve got enough to do just to with the finite dimensional ones.

The argument that shows two bases have the same size works by replacing the vectors of one with the vectors of the other. We’ll state that separately and call it the Replacement Theorem. The desired theorem, that two bases have the same number of vectors, follows from that as a corollary.

Summary of points discussed so far. Recall that a subset \( S \) of a vector space \( V \) is linearly independent if \( 0 \) is not a nontrivial linear combination of the vectors in \( S \). Equivalently, no vector in \( S \) is a linear combination of the rest of the vectors in \( S \).

Also, recall that \( S \) is a basis of \( V \) if (1) it spans \( V \), and (2) it is linearly independent. Equivalently, each vector in \( V \) may be uniquely represented as a linear combination of vectors from \( S \).

We also saw that every finite spanning subset of a vector space has a subset which is a basis of that vector space.

The Replacement Theorem. The replacement theorem will allow us to replace vectors in a basis with other vectors to get a new basis. Its proof requires attention to detail.
Continue replacing vectors until all the vectors in $T$ have replaced vectors of the original $\beta_0$. We end up with a basis of $n$ vectors that includes all $r$ vectors from $T$. Therefore $r \leq n$. Q.E.D.

**Dimension of a vector space.** The following corollary follows directly from the last theorem.

**Corollary 2.** Any two bases of a vector space have the same number of vectors (assuming at least one is a finite basis).

*Proof.* Since each basis is an independent set of vectors it has no more vectors than the other basis. Hence, the two bases have the same number of vectors. Q.E.D.

**Definition 3.** If a vector space has a finite basis, then the *dimension* of a vector space is the number of vectors in any of its bases.

For example, $\mathbb{R}^n$ has dimension $n$. Also, $P_n$, the space of polynomials in $t$ of degree less than or equal to $n$, is a vector space of dimension $n + 1$. Also, $M_{mn}$, the space of $m \times n$ matrices, has dimension $mn$.

**More on bases, linear independence, and spanning sets.** There are a few more observations we can make about bases and these related concepts.

**Corollary 4.** Any linearly independent set of vectors in a finite-dimensional vector space can be extended to a basis.

*Proof.* Take any basis and replace the vectors in that basis with the linearly independent vectors as done in the replacement theorem. You’ll end up with a basis that includes the independent set. Q.E.D.

**Corollary 5.** If $n$ vectors in a vector space of dimension $n$ are independent, then they form a basis.

*Proof.* They can be extended to a basis of $n$ vectors, but there are already $n$ vectors, so it is itself that extension. That is to say, it’s already a basis. Q.E.D.

**Corollary 6.** If $n$ vectors in a vector space of dimension $n$ span the space, then they form a basis.

*Proof.* We know that there’s a subset $S'$ of the $n$ vectors $S$ that form a basis for the space. But every basis of the space has $n$ vectors. Therefore the subset $S'$ has to be all of $S$. Q.E.D.