Elementary transformations
and matrix inversion
Math 130 Linear Algebra

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Elementary row operations again. We used the elementary row operations when we solved systems of linear equations. We'll study them more formally now, and associate each one with a particular invertible matrix.

When you want to solve a system of linear equations $A\mathbf{x} = \mathbf{b}$, form the augmented matrix by appending the column $\mathbf{b}$ to the right of the coefficient matrix A. Then solve the system of equations by operating on the rows of the augmented matrix rather than on the actual equations in the system. There are three kinds of *elementary row operations* are those operations on a matrix that don't change the solution set of the corresponding system of linear equations.

1. Exchange two rows.
2. Multiply or divide a row by a nonzero constant.
3. Add or subtract a multiple of one row from another.

With these three operations, we can convert the augmented matrix to a particularly simple form—the reduced echelon form—that allows us to read off the solutions.

Two martrices are said to be *row equivalent* if one can be transformed to the other by means of a finite sequence of elementary row operations.

Elementary matrices. Now to find the elementary matrices that correspond to these three kinds of elementary row operations. In each case, we're looking for a square matrix E such that

$$EA = B$$

where A is the augmented matrix for the original system of equations and B is the augmented matrix for the new system. In each case, we'll illustrate it with a system of three equations in three unknowns.

Let's start with this system of equations.

$$\begin{array}{ccccccc} & & 2y & + & 2z & = & 4 \\ x & & & + & 2z & = & 3 \\ & & 3y & + & 3z & = & 6 \end{array}$$

It's almost in reduced echelon form, and only three steps are needed to put it in that form. Its augmented matrix is

$$\begin{bmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & 2 & 3 \\ 0 & 3 & 3 & 6 \end{bmatrix}$$

For the first step, we'll exchange the first two rows. The elementary matrix E_1 to do that is almost the diagonal matrix. Only the two 1s in the rows to be exchanged need to be moved to the opposite rows.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 4 \\ 1 & 0 & 2 & 3 \\ 0 & 3 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 2 & 4 \\ 0 & 3 & 3 & 6 \end{bmatrix}$$

Next, let's divide the second row by 2, that is, multiply it by 0.5. Again, the elementary matrix E_2 to do that is almost the diagonal matrix. Just replace the 1 in the second row by 0.5.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 2 & 4 \\ 0 & 3 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 3 & 6 \end{bmatrix}$$

Finally, we'll subtract 3 times the second row from the third. Yet again, the elementary matrix E_3 to do that is almost the diagonal matrix. All that's needed is to place a -3 in the second column of the third row.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in reduced echelon form and the general solution can be read off from it:

$$(x, y, z) = (-2z + 3, -z + 2, z)$$

where z can be any number.

Note that each kind of elementary matrix E_1, E_2, and E_3 is invertible, and its inverse is another (or the same) elementary matrix of the same kind. The inverse for E_1 is itself. The inverse for E_2 looks just like E_2 except you use the reciprocal for the special diagonal entry. And the inverse for E_3 looks just like E_3 except you use the negation for the off-diagonal entry.

Proof that the inversion algorithm works. Recall the method used to find the inverse of a matrix.

Theorem 1. To find the inverse of a square matrix A, first, adjoin the identity matrix to its right to get an $n \times 2n$ matrix $[A|I]$. Next, convert that matrix to reduced echelon form. If the result looks like $[I|B]$, then B is the desired inverse A^{-1}. But if the square matrix in the left half of the reduced echelon form is not the identity, then A has no inverse.

Proof. Suppose the sequence of elementary row operations performed on the matrix $[A|I]$ to put it into reduced echelon form is described by the elementary matrices $E_1, E_2, \ldots, E_s$. After the first elementary operation, the matrix $[A|I]$ is converted to $[E_1A|E_1I]$. After the second, it becomes $[E_2E_1A|E_2E_1I]$. And after the last it looks like

$$[E_s \cdots E_2E_1A|E_s \cdots E_2E_1I].$$

Let's denote the product of the E_is as E, that is, $E = E_s \cdots E_2E_1$. Then we have $[EA|E]$ as the reduced echelon form.

Now, we have two cases to consider. One is where EA, the first half of the $n \times 2n$ matrix that's in reduced echelon form, has a 1 in each row. The other is where some row has no one in it but is all 0s. If any row is all 0s, then the last row will be all 0s since it's in reduced echelon form.

In the first case where there's a 1 in each row of the square matrix EA, because it's in reduced echelon form, therefore it has to be the identity. That says $EA = I$. Since E is the product $E_s \cdots E_2E_1$ of invertible elementary matrices, therefore E is also invertible. Multiply the equation $EA = I$ by E^{-1} on the left to get $A = E^{-1}$, and therefore $A^{-1} = E$. It is in this first case that A is invertible and you can find the inverse.

In the second case, the last row of EA is all 0s. We need to show that A cannot have an inverse in this case. We'll suppose A does have an inverse and derive a contradiction. Now, the matrix EA has all 0s in its last row, so if you multiply it on the right by any matrix, then the product will also have all 0s in its last row. But A is invertible, so multiply EA on the right by $A^{-1}E^{-1}$. The result is $EAA^{-1}E^{-1}I$, which is the identity matrix I, and I doesn't have 0s in its last row, a contradiction. Therefore, A doesn't have an inverse in this second case.

That finishes the proof that this method will either construct an inverse matrix or show that the inverse doesn't exist. Q.E.D.

In the proof of the theorem in the case where A is invertible, we had $A = E^{-1}$, but $E^{-1} = (E_s \cdots E_2E_1)^{-1} = E_1^{-1}E_2^{-1} \cdots E_s^{-1}$. That gives us the following corollary.

Corollary 2. Every invertible matrix is the product of elementary matrices.

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