Norm and inner products in $\mathbf{R}^n$
Math 130 Linear Algebra
D Joyce, Fall 2015

So far we've concentrated on the operations of addition and scalar multiplication in $\mathbf{R}^n$ and, more generally, in abstract vector spaces.

There are two other algebraic operations on $\mathbf{R}^n$ we mentioned early in the course, and now it's time to look at them in more detail.

One of them is the *length* of a vector, more commonly called the *norm* of a vector. The other is a kind of multiplication of two vectors called the *inner product* or *dot product* of two vectors. There's a connection between norms and inner products, and we'll look at that connection.

Today we'll restrict our discussion of these concepts to $\mathbf{R}^n$, but later we'll abstract these concepts to define *inner product spaces* in general.

The norm, or length, $\|\mathbf{v}\|$ of a vector $\mathbf{v}$. Consider a vector $\mathbf{v} = (v_1, v_2)$ in the plane $\mathbf{R}^2$. By the Pythagorean theorem of plane geometry, the distance $\|(v_1, v_2)\|$ between the point (v_1, v_2) and the origin $(0, 0)$ is

$$\|(v_1, v_2)\| = \sqrt{v_1^2 + v_2^2}.$$

Thus, we define the *length* or *norm* of a vector $\mathbf{v} = (v_1, v_2)$ as being

$$\|\mathbf{v}\| = \|(v_1, v_2)\| = \sqrt{v_1^2 + v_2^2}.$$

The norm of a vector is sometimes denoted $|\mathbf{v}|$ rather than $\|\mathbf{v}\|$.

Norms are defined for $\mathbf{R}^n$ as well

$$\begin{aligned} \|\mathbf{v}\| &= \|(v_1, v_2, \ldots, v_n)\| \\ &= \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\sum_{k=1}^{n} v_k^2}. \end{aligned}$$

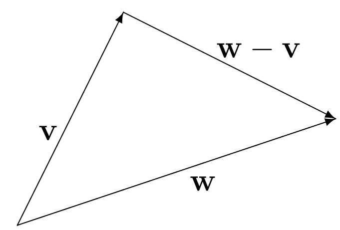
In many ways, norms act like absolute values. For instance, the norm of any vector is nonnegative, and the only vector with norm 0 is the $\mathbf{0}$ vector. Like absolute values, norms are multiplicative in the sense that

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|$$

when c is a real number and $\mathbf{v}$ is a real vector.

There's also a triangle inequality for norms

$$\|\mathbf{w} - \mathbf{v}\| \leq \|\mathbf{w}\| + \|\mathbf{v}\|.$$



Here's a geometric argument for the triangle inequality. If you draw a triangle with one side being the vector $\mathbf{v}$, and another side the vector $\mathbf{w}$, then the third side is $\mathbf{w}-\mathbf{v}$. Then the triangle inequality just says that one side of a triangle is less than or equal to the sum of the other two sides. Equality holds when the vectors $\mathbf{v}$ and $\mathbf{w}$ point in the same direction.

Later, we'll prove the triangle inequality algebraically.

The inner product $\langle \mathbf{v}|\mathbf{w}\rangle$ of two vectors. This are also commonly called a *dot product* and denoted with the alternate notation $\mathbf{v} \cdot \mathbf{w}$.

We'll start by defining inner products algebraically, then see what they mean geometrically.

The inner product $\langle \mathbf{v}|\mathbf{w}\rangle$ of two vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R}^n$ is the sum of the products of corresponding coordinates, that is,

$$\begin{aligned} \langle \mathbf{v}|\mathbf{w}\rangle &= \langle (v_1, v_2, \ldots, v_n)|(w_1, w_2, \ldots, w_n)\rangle \\ &= v_1w_1 + v_2w_2 + \cdots v_nw_n = \sum_{k=1}^{n} v_kw_k \end{aligned}$$

Notice right away that we can interpret the square of the length of the vector as an inner product. Since

$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2 + \cdots + v_n^2,$$

therefore

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}|\mathbf{v}\rangle.$$

Because of this connection between norm and inner product, we can often reduce computations involving length to simpler computations involving inner products.

The inner product acts like multiplication in a lot of ways, but not in all ways. First of all, the inner product of two vectors is a scalar, not another vector. That means you can't even ask if it's associative because the expression $\langle\langle \mathbf{u}|\mathbf{v}\rangle|\mathbf{w}\rangle$ doesn't even make sense; $\langle \mathbf{u}|\mathbf{v}\rangle$ is a scalar, so you can't take its inner product with the vector $\mathbf{w}$.

But aside from associativity, inner products act a lot like ordinary products. For instance, inner products are commutative:

$$\langle \mathbf{u}|\mathbf{v}\rangle = \langle \mathbf{v}|\mathbf{u}\rangle.$$

Also, inner products distribute over addition,

$$\langle \mathbf{u}|\mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u}|\mathbf{v}\rangle + \langle \mathbf{u}|\mathbf{w}\rangle,$$

and over subtraction,

$$\langle \mathbf{u}|\mathbf{v}-\mathbf{w}\rangle = \langle \mathbf{u}|\mathbf{v}\rangle - \langle \mathbf{u}|\mathbf{w}\rangle,$$

and the inner product of any vector and the $\mathbf{0}$ vector is 0

$$\langle \mathbf{v}|\mathbf{0}\rangle = 0.$$

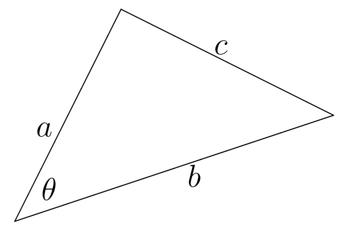
Furthermore, inner products and scalar products have a kind of associativity, namely, if c is a scalar, then

$$\langle c\mathbf{u}|\mathbf{v}\rangle = c\langle \mathbf{u}|\mathbf{v}\rangle = \langle \mathbf{u}|c\mathbf{v}\rangle.$$

These last few statements can be summarized by saying that inner products are linear in each coordinate, or that inner products are bilinear operations.

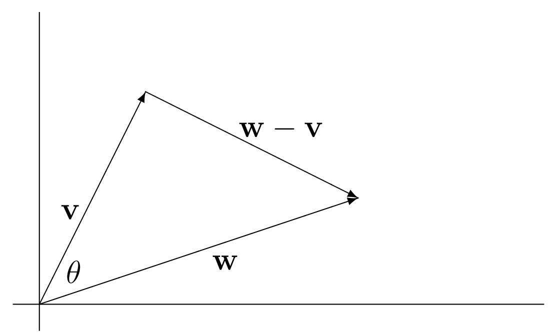
The inner product of two vectors and the cosine of the angle between them. For this discussion, we'll restrict our attention to dimension 2 since we know a lot of plane geometry.

The law of cosines for oblique triangles says that given a triangle with sides a, b, and c, and angle θ between sides a and b,



$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Now, start with two vectors $\mathbf{v}$ and $\mathbf{w}$, and place them in the plane with their tails at the same point. Let θ be the angle between these two vectors. The vector that joins the head of $\mathbf{v}$ to the head of $\mathbf{w}$ is $\mathbf{w}-\mathbf{v}$. Now we can use the law of cosines to see that



$$\|\mathbf{w}-\mathbf{v}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta.$$

We can convert the distances to inner products to simplify this equation.

$$\begin{aligned}\|\mathbf{w}-\mathbf{v}\|^2 &= \langle \mathbf{w}-\mathbf{v}|\mathbf{w}-\mathbf{v}\rangle \\ &= \langle \mathbf{w}|\mathbf{w}\rangle - 2\langle \mathbf{w}|\mathbf{v}\rangle + \langle \mathbf{v}|\mathbf{v}\rangle \\ &= \|\mathbf{w}\|^2 - 2\langle \mathbf{w}|\mathbf{v}\rangle + \|\mathbf{v}\|^2\end{aligned}$$

Now, if we subtract $\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$ from both sides of our equation, and then divide by -2, we get

$$\langle \mathbf{v}|\mathbf{w}\rangle = \|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta.$$

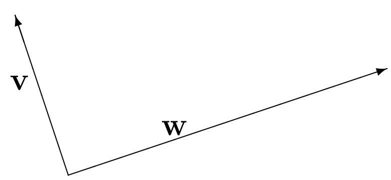
That gives us a way of geometrically interpreting the inner product. We can also solve the last equation for $\cos\theta$,

$$\cos\theta = \frac{\langle \mathbf{v}|\mathbf{w}\rangle}{\|\mathbf{v}\|\,\|\mathbf{w}\|},$$

which will allow us to do trigonometry by means of linear algebra. Note that

$$\theta = \arccos\left(\frac{\langle \mathbf{v}|\mathbf{w}\rangle}{\|\mathbf{v}\|\,\|\mathbf{w}\|}\right).$$

Orthogonal vectors. The word "orthogonal" is synonymous with the word "perpendicular," but for some reason is preferred in many branches of mathematics. We'll write $\mathbf{w} \perp \mathbf{v}$ if the vectors $\mathbf{w}$ and $\mathbf{v}$ are orthogonal, or perpendicular.



Two vectors are orthogonal if the angle between them is 90°. Since the cosine of 90° is 0, that means

$$\mathbf{w} \perp \mathbf{v} \text{ if and only if } \langle \mathbf{w}|\mathbf{v}\rangle = 0$$

Vectors in Matlab**.** You can easily find the length of a vector in Matlab; where the length of a vector is called its `norm`. Let's find the length of two vectors and the angle between them using the formula

$$\theta = \arccos\left(\frac{\langle \mathbf{v}|\mathbf{w}\rangle}{\|\mathbf{v}\|\,\|\mathbf{w}\|}\right).$$

Note that arccosines are computed with the `acos` function, and inner products with the dot function

```
>> u = [3 4]

u =
     3     4

>> norm(u)

ans =
     5

>> v = [5 12]

v =
     5    12

>> norm (v)

ans =
    13

>> dot(u,v)

ans =
    63

>> costheta = dot(u,v)/(norm(u)*norm(v))

costheta =
    0.9692

>> acos(costheta)

ans =
    0.2487
```

Thus, the angle between the vectors $(3, 4)$ and $(5, 12)$ is 0.2487 radians.

Math 130 Home Page at
`http://math.clarku.edu/~ma130/`