Applications of inner products in $\mathbf{R}^n$
Math 130 Linear Algebra
D Joyce, Fall 2015

Summary of norms and inner products in $\mathbf{R}^n$ discussed last time.

The norm, or length, $\|\mathbf{v}\|$ of a vector $\mathbf{v}$ is

$$\|\mathbf{v}\| = \|(v_1, v_2, \ldots, v_n)\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

The inner product $\langle \mathbf{v}|\mathbf{w}\rangle$ of two vectors is

$$\begin{aligned} \langle \mathbf{v}|\mathbf{w}\rangle &= \langle (v_1, v_2, \ldots, v_n)|(w_1, w_2, \ldots, w_n)\rangle \\ &= v_1 w_1 + v_2 w_2 + \cdots + v_n w_n \end{aligned}$$

Norms can be written in terms of inner products

$$\|\mathbf{v}\|^2 = \langle \mathbf{v}|\mathbf{v}\rangle.$$

The law of cosines can be incorporated into linear algebra as

$$\langle \mathbf{v}|\mathbf{w}\rangle = \|\mathbf{v}\|\,\|\mathbf{w}\|\cos\theta.$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. A special case of this when θ is a right angle gives us

$$\mathbf{w} \perp \mathbf{v} \text{ if and only if } \langle \mathbf{w}|\mathbf{v}\rangle = 0.$$

Unit vectors. A *unit vector* is a vector whose length is 1. Now that we can talk about the length of a vector, we can construct a unit vector in the same direction as a given vector simply by dividing by its length. If $\mathbf{v}$ is a vector in $\mathbf{R}^n$, the unit vector in the same direction is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

In $\mathbf{R}^2$, if a unit vector $\mathbf{u}$ is placed in standard position with its tail at the origin, then its head will land on the unit circle $x^2+y^2 = 1$. Every point on the unit circle (x, y) is of the form $(\cos\theta, \sin\theta)$ where θ is the angle measured from the positive x-axis in the counterclockwise direction.

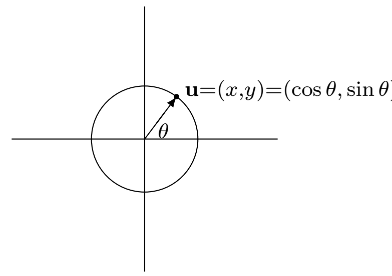

Figure 1: Unit vectors in $\mathbf{R}^2$

Thus, every unit vector in the plane is of the form

$$\mathbf{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

We can interpret unit vectors as being directions, and we can use them in place of angles since they carry the same information as an angle.

When we move on to three dimensions, we'll still use unit vectors and they will still signify directions, but they carry more information than just one angle. After all, if you want to name a point on a sphere, you need to give two angles, longitude and latitude.

Now that we have unit vectors, we can treat every vector $\mathbf{v}$ as a length and a direction. The length of $\mathbf{v}$ is $\|\mathbf{v}\|$, of course. And its direction is the unit vector $\mathbf{u}$ in the same direction which can be found by

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

The vector $\mathbf{v}$ can be reconstituted from its length and direction by multiplying $\mathbf{v} = \|\mathbf{v}\|\,\mathbf{u}$.

Parametric descriptions for lines in the plane and in space. A useful way to describe a line, either in the plane or in space, is to name one point $\mathbf{a}$ on the line and say that other points on the line

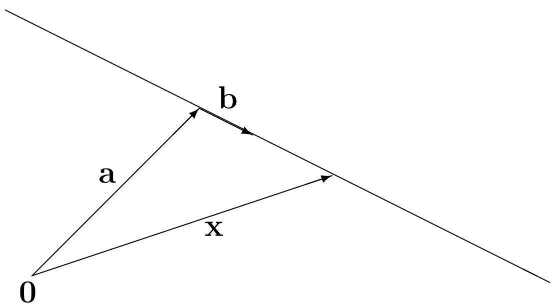

Figure 2: Parameterized line $\mathbf{x} = \mathbf{a} + t\mathbf{b}$

can be found by going some distance in a direction $\mathbf{b}$ from that point. Then a typical point on the line is of the form

$$\mathbf{x} = \mathbf{a} + t\mathbf{b}$$

where t is any real number. Here, we say that t is a parameter for describing points on the line. The vector $\mathbf{b}$ is often taken to be a unit vector, but not always.

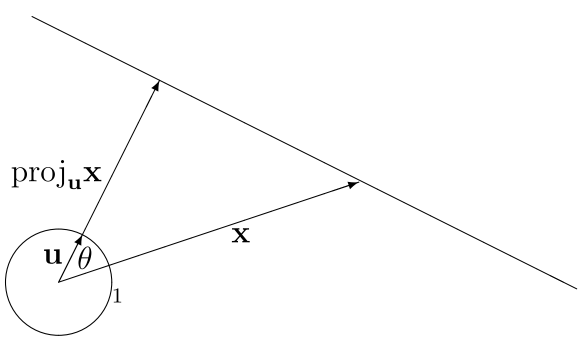

Figure 3: Projection onto a unit vector $\mathbf{u}$

Projections of one vector onto another. In general, $\langle \mathbf{a}|\mathbf{x}\rangle = \|\mathbf{a}\|\,\|\mathbf{x}\| \cos\theta$, where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{x}$, but when $\mathbf{a}$ is $\mathbf{u}$, a unit vector, that equation simplifies to $\mathbf{u} \cdot \mathbf{x} = \|\mathbf{x}\| \cos\theta$. We can interpret this quantity as a distance as follows. Project the vector $\mathbf{x}$ on to the unit vector $\mathbf{u}$. We get a right triangle with one side being that projection, and the hypotenuse $\mathbf{x}$. The projection has length $\|\mathbf{x}\| \cos\theta$. Thus , $\mathbf{u} \cdot \mathbf{x}$ is the length of the projection of $\mathbf{x}$ on to $\mathbf{u}$. The projection of the vector $\mathbf{x}$ onto the vector $\mathbf{u}$ is a vector denoted as $\text{proj}_{\mathbf{u}}\mathbf{x}$. Thus we have

$$\text{proj}_{\mathbf{u}}\mathbf{x} = \langle \mathbf{x}|\mathbf{u}\rangle\mathbf{u} = (\|\mathbf{x}\| \cos\theta)\mathbf{u}.$$

The projection of $\mathbf{x}$ onto the vector $\mathbf{a}$ is the same vector, $\text{proj}_{\mathbf{a}}\mathbf{x} = \text{proj}_{\mathbf{u}}\mathbf{x}$, where $\mathbf{u} = \dfrac{\mathbf{a}}{\|\mathbf{a}\|}$. Thus

$$\begin{aligned} \text{proj}_{\mathbf{a}}\mathbf{x} &= \langle \mathbf{x}|\mathbf{u}\rangle\mathbf{u} \\ &= \left\langle \mathbf{x} \middle| \frac{\mathbf{a}}{\|\mathbf{a}\|} \right\rangle \frac{\mathbf{a}}{\|\mathbf{a}\|} \\ &= \frac{\langle \mathbf{x}|\mathbf{a}\rangle}{\|\mathbf{a}\|^2}\mathbf{a} \end{aligned}$$

This projection of one vector on another is useful in answering some questions about geometry using linear algebra. It works in all dimensions.

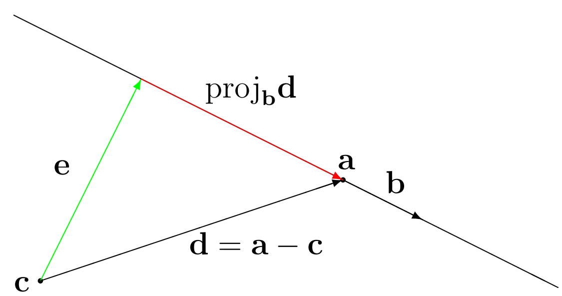

Figure 4: Distance from $\mathbf{c}$ to $\mathbf{a} + t\mathbf{b}$

Distance from a point to a line. Given a point $\mathbf{c}$ and a line $\mathbf{x} = \mathbf{a} + t\mathbf{b}$, how can you find the distance from the point to the line?

Refer to figure 4. Treat $\mathbf{c}$ as a point and $\mathbf{a}$ as a point, but $\mathbf{b}$ as an arrow, the direction of the line. The displacement vector $\mathbf{d} = \mathbf{a} - \mathbf{c}$ goes from the point $\mathbf{c}$ to the point $\mathbf{a}$. Project it onto the line to get the red arrow $\text{proj}_{\mathbf{b}}\mathbf{d}$. Subtract that from $\mathbf{d}$ to get the vector $\mathbf{e}$, shown in green, from the point $\mathbf{c}$ to the point on the line closest to $\mathbf{c}$. The length of $\mathbf{e}$ is the distance from the point to the line.

The formulas and computations may simplify a bit when $\mathbf{b}$ is a unit vector.

Example 1. How far is it from the point $\mathbf{c} = (1, 2, 3)$ to the line $\mathbf{x} = \mathbf{a} + t\mathbf{b} = (4, 4, 4) + t(1, 1, 1)$?

Let $\mathbf{d} = \mathbf{a} - \mathbf{c} = (3, 2, 1)$. Then

$$\langle \mathbf{d} | \mathbf{b} \rangle = \langle (3, 2, 1) | (1, 1, 1) \rangle = 6,$$

and $\|\mathbf{b}\|^2 = \|(1, 1, 1)\|^2 = 3$, so

$$\mathrm{proj}_{\mathbf{b}} \mathbf{d} = \frac{\langle \mathbf{d} | \mathbf{b} \rangle}{\|\mathbf{b}\|^2} \mathbf{b} = 2\mathbf{b} = (2, 2, 2).$$

Thus, $\mathbf{e} = \mathbf{d} - \mathrm{proj}_{\mathbf{b}} \mathbf{d} = (1, 0, -1)$. Therefore, the distance from the point to the line is $\|\mathbf{e}\| = \|(1, 0, -1)\| = \sqrt{2}$.

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