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Matrix inversion
Math 130 Linear Algebra

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Recall that a linear transformation T is an isomorphism when it has an inverse, that is, there's another linear transformation S such that $T \circ S$ and $S \circ T$ are both the identity transformation.

As we've noted before, two finite dimensional vector spaces can be isomorphic if and only if they have the same dimension. We'll denote that dimension by n.

Matrices encode linear transformations on finite dimensional vector spaces, so we can translate that statement to define invertibility of matrices.

Definition 1. We say that two square $n \times n$ matrices A and B are *inverses* of each other if

$$AB = BA = I$$

and in that case we say that B is an inverse of A and that A is an inverse of B. If a matrix has no inverse, it is said to be *singular*, but if it does have an inverse, it is said to be *invertible* or *nonsingular*.

With that definition, a matrix is invertible if and only if it represents an invertible linear transformation $F^n \xrightarrow{\cong} F^n$, in which case its inverse represents the inverse linear transformation.

A linear transformation T can have at most one inverse, which we denote T^{-1}, and $(T^{-1})^{-1} = T$. Therefore, we can conclude the following theorem.

Theorem 2. A matrix A can have at most one inverse. The inverse of an invertible matrix is denoted A^{-1}. Also, when a matrix is invertible, so is its inverse, and its inverse's inverse is itself, $(A^{-1})^{-1} = A$.

Note that inversion reverses the order of composition. If S and T are isomorphisms, then so is $S \circ T$, and $(S \circ T)^{-1} = (T^{-1}) \circ (S^{-1})$.

$$\begin{tikzcd}
V \arrow[r, shift right, "T"'] \arrow[dr, shift left, "S \circ T"] & W \arrow[l, shift right, "T^{-1}"'] \arrow[d, shift right, "S"'] \\
& X \arrow[u, shift right, "S^{-1}"'] \arrow[ul, shift left, "(S \circ T)^{-1}"]
\end{tikzcd}$$

We can translate that into the language of matrices as the following theorem.

Theorem 3. If A and B are both invertible, then their product is, too, and $(AB)^{-1} = B^{-1}A^{-1}$.

Inverses of 2×2 matrices. You can easily find the inverse of a 2×2 matrix. Consider a generic 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

It's inverse is the matrix

$$A^{-1} = \begin{bmatrix} d/\Delta & -b/\Delta \\ -c/\Delta & a/\Delta \end{bmatrix}$$

where Δ is the determinant of A, namely

$$\Delta = ad - bc,$$

provided Δ is not 0. In words, to find the inverse of a 2×2 matrix, (1) exchange the entries on the major diagonal, (2) negate the entries on the minor diagonal, and (3) divide all four entries by the determinant.

It's easy to verify that A^{-1} actually is the inverse of A, just multiply them together to get the identity matrix I.

A method for finding inverse matrices. Next we'll look at a different method to determine if an $n \times n$ square matrix A is invertible, and if it is what it's inverse is.

The method is this. First, adjoin the identity matrix to its right to get an $n \times 2n$ matrix $[A|I]$. Next, convert that matrix to reduced echelon form. If the result looks like $[I|B]$, then B is the desired inverse A^{-1}. But if the square matrix in the left half of the reduced echelon form is not the identity, then A has no inverse.

We'll verify that this method works later.

Example 4. Let's illustrate the method with a 3×3 example. Let A be the matrix

$$A = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

Form the 3×6 matrix $[A|I]$, and row reduce it. I'll use the symbol $\sim$ when a row-operation is applied. Here are the steps.

$$\begin{aligned}
[A|I] &= \left[\begin{array}{rrr|rrr} 3 & -2 & 4 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \\
&\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 3 & -2 & 4 & 1 & 0 & 0 \end{array}\right] \\
&\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & 1 & -3 & 0 \end{array}\right] \\
&\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & -3 & 2 \end{array}\right] \\
&\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1/2 & 3/2 & -1 \end{array}\right] = [I|A^{-1}]
\end{aligned}$$

This row-reduction to reduced echelon form succeeded in turning the left half of the matrix into the identity matrix. When that happens, the right half of the matrix will be the inverse matrix A^{-1}. Therefore, the inverse matrix is

$$A^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \\ -1/2 & 3/2 & -1 \end{bmatrix}.$$

Matlab **can compute inverses** or tell you if they're singular.

```
>> A = [1 2; 3 4]
A =
     1     2
     3     4

>> B = inv(A)
B =
   -2.0000    1.0000
    1.5000   -0.5000

>> A*B
ans =
    1.0000         0
    0.0000    1.0000

>> C = [1 2; 3 6]
C =
     1     2
     3     6

>> D = inv(C)
Warning: Matrix is singular to working precision.
D =
   Inf   Inf
   Inf   Inf
```

Math 130 Home Page at
http://math.clarku.edu/~ma130/