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Notes on Quaternions
Math 130 Linear Algebra

D Joyce, Fall 2015

> Sir William Rowan Hamilton, who early found that his road [to success with vectors] was obstructed—he knew not by what obstacle—so that many points which seemed within his reach were really inaccessible. He had done a considerable amount of good work, obstructed as he was, when, about the year 1843, he perceived clearly the obstruction to his progress in the shape of an old law which, prior to that time, had appeared like a law of common sense. The law in question is known as the *commutative* law of multiplication.
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> P. Kelland (1808–1879) and P. G. Tait (1831–1901), 1873

The quaternions. The quaternions $\mathbf{H}$ are an extension of the complex numbers $\mathbf{C}$. A complex number can be defined as an expression $a+bi$ where a and b are real numbers and i is a formal symbol satisfying $i^2=-1$. We'll add two more formal symbols, j and k to define $\mathbf{H}$.

We can define a quaternion a as an expression

$$a = a_0 + a_1 i + a_2 j + a_3 k$$

where a_0, a_1, a_2, and a_3 are real numbers and i, j, and k are formal symbols satisfying the properties

$$i^2 = j^2 = k^2 = -1$$

and

$$ij = k, jk = i, ki = j.$$

The i, j, and k are all square roots of -1, but they don't commute as you can show from the definition that

$$ji = -k, kj = -i, ik = -j.$$

This doesn't lead to a commutative multiplication, but note that if a is real (i.e., its pure quaternion parts a_1, a_2, and a_3 are all 0), then a will commute with any quaternion b.

The quaternions are collectively denoted $\mathbf{H}$ in honor of Hamilton, their inventor.

Addition and subtraction are coordinatewise (just like in the complex numbers $\mathbf{C}$).

$$(a_0 + a_1 i + a_2 j + a_3 k) \pm (b_0 + b_1 i + b_2 j + b_3 k) = (a_0 \pm b_0) + (a_1 \pm b_1)i + (a_2 \pm b_2)j + (a_3 \pm b_3)k$$

Here's multiplication.

$$\begin{aligned}
& (a_0 + a_1 i + a_2 j + a_3 k)\,(b_0 + b_1 i + b_2 j + b_3 k) \\
= \;& (a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3) \\
+ \;& (a_0 b_1 + a_1 b_0 + a_2 b_3 - a_3 b_2) i \\
+ \;& (a_0 b_2 - a_1 b_3 + a_2 b_0 + a_3 b_1) j \\
+ \;& (a_0 b_3 + a_1 b_2 - a_2 b_1 - a_3 b_0) k
\end{aligned}$$

All the usual properties of addition, subtraction, and multiplication hold in $\mathbf{H}$ except one. Multiplication in anticommutative; if a and b are two pure quaternions, that is, quaternions with no real part then

$$ab = -ba.$$

Division works, too; reciprocals exist for all quaternions except 0. We can use a variant of rationalizing the denominator to find the reciprocal of a quaternion.

$$\begin{aligned}
\frac{1}{a_0 + a_1 i + a_2 j + a_3 k} &= \frac{a_0 - a_1 i - a_2 j - a_3 k}{(a_0 - a_1 i - a_2 j - a_3 k)(a_0 + a_1 i + a_2 j + a_3 k)} \\
&= \frac{a_0 - a_1 i - a_2 j - a_3 k}{a_0^2 + a_1^2 + a_2^2 + a_3^2}
\end{aligned}$$

Thus, a nonzero quaternion $a_0 + a_1 i + a_2 j + a_3 k$, that is, one where not all of the real numbers a_0, a_1, a_2, and a_3 are 0, has an inverse, since the denominator $a_0^2 + a_1^2 + a_2^2 + a_3^2$ is a nonzero real number.

The expression $a_0 - a_1 i - a_2 j - a_3 k$ used to rationalize the denominator is the *conjugate* of the original quaternion $a_0 + a_1 i + a_2 j + a_3 k$. The standard notation for denoting the conjugate of a quaternion is to place a bar over it. Thus

$$\overline{a_0 + a_1 i + a_2 j + a_3 k} = a_0 - a_1 i - a_2 j - a_3 k.$$

The *norm* or *absolute value* of a quaternion a by $|a|^2 = a\overline{a}$. It's a nonnegative real number, so it has a square root $|a|$.

Thus, if a is a nonzero quaternion, then its inverse is

$$\frac{1}{a} = \frac{\overline{a}}{|a|^2}.$$

Note that the norm of a product is the product of the norms since

$$|ab|^2 = ab\overline{ab} = ab\overline{b}\overline{a} = a|b|^2\overline{a} = a\overline{a}|b|^2 = |a|^2\,|b|^2.$$

Quaternions and geometry. Each quaternion a is the sum of a real part a_0 and a pure quaternion part $a_1 i + a_2 j + a_3 k$. Hamilton called the real part a *scalar* and pure quaternion part a *vector*. We can interpret $a_1 i + a_2 j + a_3 k$ as a vector $\mathbf{a} = (a_1, a_2, a_3)$ in $\mathbf{R}^3$. Addition and subtraction of pure quaternions then are just ordinary vector addition and subtraction.

Hamilton recognized that the product of two vectors (pure quaternions) had both a vector component and a scalar component (the real part). The vector component of the product $\mathbf{ab}$ of two pure quaternions Hamilton called the *vector product*, now often denoted $\mathbf{a} \times \mathbf{b}$ or

$\mathbf{a} \wedge \mathbf{b}$, and called the *cross product* or the *outer product*. The negation of the scalar component Hamilton called the *scalar product*, now often denoted $\mathbf{a} \cdot \mathbf{b}$, $(\mathbf{a}, \mathbf{b})$, $\langle \mathbf{a}, \mathbf{b} \rangle$, or $\langle \mathbf{a} | \mathbf{b} \rangle$ and called the *dot product* or the *inner product*. Thus

$$\mathbf{ab} = \mathbf{a} \times \mathbf{b} - \mathbf{a} \cdot \mathbf{b}.$$

Hamilton's quaternions were very successful in the 19th century in the study of three-dimensional geometry.

Here's a typical problem from Kelland and Tait's 1873 *Introduction to Quaternions*. If three mutually perpendicular vectors be drawn from a point to a plane, the sum of the reciprocals of the squares of their lengths is independent of their directions.

Matrices were invented later in the 19th century. (But determinants were invented earlier!) Matrix algebra supplanted quaternion algebra in the early 20th century because (1) they described linear transformations, and (2) they weren't restricted to three dimensions.

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