CLARK UNIVERSITY

Kernel, image, nullity, and rank
Math 130 Linear Algebra

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Definition 1. Let $T : V \to W$ be a linear transformation between vector spaces. The *kernel* of T, also called the *null space* of T, is the inverse image of the zero vector, $\mathbf{0}$, of W,

$$\ker(T) = T^{-1}(\mathbf{0}) = \{\mathbf{v} \in V \mid T\mathbf{v} = \mathbf{0}\}.$$

It's sometimes denoted $N(T)$ for null space of T.

The *image* of T, also called the *range* of T, is the set of values of T,

$$T(V) = \{T(\mathbf{v}) \in W \mid \mathbf{v} \in V\}.$$

This image is also denoted $\mathrm{im}(T)$, or $R(T)$ for range of T.

Both of these are vector spaces. $\ker(T)$ is a subspace of V, and $T(V)$ is a subspace of W. (Why? Prove it.)

We can prove something about kernels and images directly from their definition.

Theorem 2. Let $V \xrightarrow{T} W$ and $W \xrightarrow{U} X$ be linear transformations. Then

$$\ker(T) \subseteq \ker(U \circ T)$$

and

$$\mathrm{im}(U \circ T) \subseteq \mathrm{im}(U).$$

Proof. For the first statement, just note that $T(\mathbf{v}) = \mathbf{0}$ implies $U(T(\mathbf{v})) = \mathbf{0}$. For the second statement, just note that an element of the form $U(T(\mathbf{v})$ in X is automatically of the form $U(\mathbf{w})$ where $\mathbf{w} = T(\mathbf{v})$. Q.E.D.

Definition 3. The dimensions of the kernel and image of a transformation T are called the transformation's *rank* and *nullity*, and they're denoted $\mathrm{rank}(T)$ and $\mathrm{nullity}(T)$, respectively. Since a matrix represents a transformation, a matrix also has a *rank* and *nullity*.

For the time being, we'll look at ranks and nullity of transformations. We'll come back to these topics again when we interpret our results for matrices.

The above theorem implies this corollary.

Corollary 4. Let $V \xrightarrow{T} W$ and $W \xrightarrow{U} X$. Then

$$\mathrm{nullity}(T) \leq \mathrm{nullity}(U \circ T)$$

and

$$\mathrm{rank}(U \circ T) \leq \mathrm{rank}(U).$$

Systems of linear equations and linear transformations. We've seen how a system of m linear equations in n unknowns can be interpreted as a single matrix equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x}$ is the $n \times 1$ column vector whose entries are the n unknowns, and $\mathbf{b}$ is the $m \times 1$ column vector of constants on the right sides of the m equations.

We can also interpret a system of linear equations in terms of a linear transformation. Let the linear transformation $T : \mathbf{R}^n \to \mathbf{R}^m$ correspond to the matrix A, that is, $T(\mathbf{x}) = A\mathbf{x}$. Then the matrix equation $A\mathbf{x} = \mathbf{b}$ becomes

$$T(\mathbf{x}) = \mathbf{b}.$$

Solving the equation means looking for a vector $\mathbf{x}$ in the inverse image $T^{-1}(\mathbf{b})$. It will exist if and only if $\mathbf{b}$ is in the image $T(V)$.

When the system of linear equations is homogeneous, then $\mathbf{b} = \mathbf{0}$. Then the solution set is the subspace of V we've called the kernel of T. Thus, kernels are solutions to homogeneous linear equations.

When the system is not homogeneous, then the solution set is not a subspace of V since it doesn't contain $\mathbf{0}$. In fact, it will be empty when $\mathbf{b}$ is not in the image of T. If it is in the image, however, there is at least one solution $\mathbf{a} \in V$ with $T(\mathbf{a}) = \mathbf{b}$. All the rest can be found from $\mathbf{a}$ by adding solutions $\mathbf{x}$ of the associated homogeneous equations, that is,

$$T(\mathbf{a}+\mathbf{x}) = \mathbf{b} \quad \text{iff} \quad T(\mathbf{x}) = \mathbf{0}.$$

Geometrically, the solution set is a translate of the kernel of T, which is a subspace of V, by the vector $\mathbf{a}$.

Example 5. Consider this nonhomogeneous linear system $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solve it by row reducing the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & 1 \\ 1 & 1 & 3 & 2 \end{array}\right]$$

to

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 5/2 & 3/2 \end{array}\right]$$

Then z can be chosen freely, and x and y determined from z, that is,

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}z \\ \frac{3}{2} - \frac{5}{2}z \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \end{bmatrix} z$$

This last equation is the parametric equation of a line in $\mathbf{R}^3$, that is to say, the solution set is a line. But it's not a line through the origin. There is, however, a line through the origin, namely

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{5}{2} \\ 1 \end{bmatrix} z$$

and that line is the solution space of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. Furthermore, these two lines are parallel, and the vector $\begin{bmatrix} 1/2 \\ 3/2 \\ 0 \end{bmatrix}$ shifts the line through the origin to the other line. In summary, for this example, the solution set for the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is a line in $\mathbf{R}^3$ parallel to the solution space for the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

In general, the solution set for the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ won't be a one-dimensional line. It's dimension will be the nullity of A, and it will be parallel to the solution space of the associated homogeneous equation $A\mathbf{x} = \mathbf{0}$.

The dimension theorem. The rank and nullity of a transformation are related. Specifically, their sum is the dimension of the domain of the transformation. That equation is sometimes called the dimension theorem.

In terms of matrices, this connection can be stated as the rank of a matrix plus its nullity equals the number of rows of the matrix.

Before we prove the Dimension Theorem, first we'll find a characterization of the image of a transformation.

Theorem 6. The image of a transformation is spanned by the image of the any basis of its domain. For $T : V \to W$, if $\beta = \{\mathbf{b}_1, \mathbf{b_2}, \ldots, \mathbf{b_n}\}$ is a basis of V, then $T(\beta) = \{T(\mathbf{b}_1), T(\mathbf{b_2}), \ldots, T(\mathbf{b_n})\}$ spans the image of T.

Although $T(\beta)$ spans the image, it needn't be a basis because its vectors needn't be independent.

Proof. A vector in the image of T is of the form $T(\mathbf{v})$ where $\mathbf{v} \in V$. But β is a basis of V, so $\mathbf{v}$ is a linear combination of the basis vectors $\{\mathbf{b}_1, \ldots, \mathbf{b_n}\}$. Therefore, $T(\mathbf{v})$ is same linear combination of the vectors $T(\mathbf{b}_1), \ldots, T(\mathbf{b_n})$. Therefore, every vector in the image of T is a linear combination of vectors in $T(\beta)$. Q.E.D.

Theorem 7 (Dimension Theorem)**.** If the domain of a linear transformation is finite dimensional, then that dimension is the sum of the rank and nullity of the transformation.

Proof. Let $T : V \to W$ be a linear transformation, let n be the dimension of V, let r be the rank of T and k the nullity of T. We'll show $n = r + k$.

Let $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ be a basis of the kernel of T. This basis can be extended to a basis $\gamma =$

$\{\mathbf{b}_1, \ldots, \mathbf{b}_k, \ldots, \mathbf{b}_n\}$ of all of V. We'll show that the image of the vectors we appended,

$$C = \{T(\mathbf{b}_{k+1}), \ldots, T(\mathbf{b}_n)\}$$

is a basis of $T(V)$. That will show that $r = n - k$ as required.

First, we need to show that the set C spans the image $T(V)$. From the previous theorem, we know that $T(\gamma)$ spans $T(V)$. But all the vectors in $T(\beta)$ are $\mathbf{0}$, so they don't help in spanning $T(V)$. That leaves $C = T(\gamma) - T(\beta)$ to span $T(V)$.

Next, we need to show that the vectors in C are linearly independent. Suppose that $\mathbf{0}$ is a linear combination of them,

$$c_{k+1}T(\mathbf{b}_{k+1}) + \cdots + c_nT(\mathbf{b}_n) = \mathbf{0}$$

where the c_i's are scalars. Then

$$T(c_{k+1}\mathbf{b}_{k+1} + \cdots + b_n\mathbf{v}_n) = \mathbf{0}$$

Therefore, $\mathbf{v} = c_{k+1}\mathbf{b}_{k+1} + \cdots + c_n\mathbf{b}_n$ lies in the kernel of T. Therefore, $\mathbf{v}$ is a linear combination of the basis vectors β, $\mathbf{v} = c_0\mathbf{b}_0 + \cdots + c_k\mathbf{b}_k$. These last two equations imply that $\mathbf{0}$ is a linear combination of the entire basis γ of V,

$$c_0\mathbf{b}_0 + \cdots + c_k\mathbf{b}_k - c_{k+1}\mathbf{b}_{k+1} - \cdots - c_n\mathbf{b}_n = \mathbf{0}.$$

Therefore, all the coefficients c_i are 0. Therefore, the vectors in C are linearly independent.

Thus, C is a basis of the image of T. Q.E.D.

There are several results that follow from this Dimension Theorem.

Characterization of one-to-one transformations. Recall that a function f is said to be a *one-to-one* function if whenever $x \neq y$ then $f(x) \neq f(y)$. That applies to transformations as well. $T : V \to W$ is one-to-one when $\mathbf{u} \neq \mathbf{v}$ implies $T(\mathbf{u}) \neq T(\mathbf{v})$. An alternate term for one-to-one is *injective*, and yet another term that's often used for transformations is *monomorphism*.

Which transformations are one-to-one can be determined by their kernels.

Theorem 8. A transformation is one-to-one if and only if its kernel is trivial, that is, its nullity is 0.

Proof. Let $T : V \to W$. Suppose that T is one-to-one. Then since $T(\mathbf{0}) = \mathbf{0}$, therefore T can send no other vector to $\mathbf{0}$. Thus, the kernel of T consists of $\mathbf{0}$ alone. That's a 0-dimensional space, so the nullity of T is 0.

Conversely, suppose that the nullity of T is 0, that is, its kernel consists only of $\mathbf{0}$. We'll show T is one-to-one. Suppose that $\mathbf{u} \neq \mathbf{v}$. Then $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$. Then $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ does not lie in the kernel of T which means that $T(\mathbf{u} - \mathbf{v}) \neq 0$. But $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$, therefore $T(\mathbf{u}) \neq T(\mathbf{v})$. Thus, T is one-to-one. Q.E.D.

Since the rank plus the nullity of a transformation equals the dimension of its domain, $r + k = n$, we have the following corollary.

Corollary 9. For a transformation T whose domain is finite dimensional, T is one-to-one if and only if that dimension equals its rank.

Characterization of isomorphisms. If two vector spaces V and W are isomorphic, their properties with respect to addition and scalar multiplication will be the same. A subset of V will span V if and only if its image in W spans W. It's independent in V if and only if its image is independent in W. Therefore, an isomorphism sends any basis of the first to a basis of the second. And that means

Theorem 10. Isomorphic vector spaces have the same dimension.

We'll be interested in which linear transformations are isomorphisms. The Dimension Theorem gives us some criteria for that.

Theorem 11. Let $T : V \to W$ be a linear transformation be a transformation between two vector spaces of the same dimension. Then the following statements are equivalent.

(1) T is an isomorphism.
(2) T is one-to-one.

(3) The nullity of T is 0, that is, its kernel is trivial.

(4) T is onto, that is, the image of T is W.

(5) The rank of T is the same as the dimension of the vector spaces.

Proof. (2) and (3) are equivalent by a previous theorem. (4) and (5) are equivalent by the definition of rank. Since $n = r + k$, (3) is equivalent to (5). (1) is equivalent to the conjunction of (2) and (3), but they're equivalent to each other, so they're equivalent to (1). Q.E.D.

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