



Span and independence  
Math 130 Linear Algebra  
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We're looking at bases of vector spaces. Recall that a basis  $\beta$  of a vector space  $V$  is a set of vectors  $\beta = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n)$  such that each vector  $\mathbf{v}$  in  $V$  can be uniquely represented as a linear combination of vectors from  $\beta$

$$\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \dots + v_n\mathbf{b}_n.$$

How can a set  $S$  of vectors fail to be a basis for a vector space  $V$ ? There are two ways. It might be that some vectors aren't linear combinations of  $S$ , that is, there aren't enough vectors to span all of  $V$ . It might be that some vectors can be expressed as linear combinations of  $S$  but in more than one way, that is, there are too many vectors in  $S$ . We'll study these two phenomena next.

**The span of a set of vectors.** Since vector spaces are closed under linear combinations, we should have a name for the set of all linear combinations of a given set of vectors, and that will be their *span*.

**Definition 1.** Let  $S$  be a set of vectors in a vector space  $V$ . The *span* of  $S$ , written  $\text{span}(S)$ , is the set of all linear combinations of vectors in  $S$ . That is,  $\text{span}(S)$  consists of all vectors of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_k$$

where each  $c_i$  is a scalar and each  $\mathbf{v}_i$  is a vector in  $S$ .

The proof of the following theorem is left for you to prove. It depends on showing that a linear combination of linear combinations is a linear combination.

**Theorem 2.** The span of a set  $S$  is a subspace of  $V$ .

You can also describe  $\text{span}(S)$  as the smallest subspace of  $V$  that contains all of  $S$ .

**Theorem 3.** The span of a set  $S$  is the intersection of all subspaces of  $V$  that contain  $S$ .

$$\text{span}(S) = \bigcap \{W, \text{ a subspace of } V \mid S \subseteq W\}$$

*Proof.* First note that  $\text{span}(S)$  is a vector space that contains all of  $S$ , so it's one of spaces  $W$  in the intersection. Second,  $\text{span}(S)$  only has linear combinations of vectors in  $S$ , so every vector in  $\text{span}(S)$  has to be in every vector space  $W$  that contains all of  $S$ . Therefore  $\text{span}(S)$  is a subset of all the spaces  $W$  in the intersection, so it's the smallest one, and, therefore, equals the intersection of all of them. Q.E.D.

Some examples. A single nontrivial vector in  $\mathbf{R}^n$  spans the line through the origin that contains it. Two vectors in  $\mathbf{R}^3$  that don't both lie in the same line span a plane. The functions  $\sin t$  and  $\cos t$  span the solution space of the differential equation  $y'' = -y$ .

**Definition 4.** We say a set  $S$  of vectors in a vector space  $V$  spans  $V$  if  $V = \text{span}(S)$ . Equivalently, every vector in  $V$  is a linear combination of vectors in  $S$ .

Note that this definition does not require that a vector can be a linear combination in only one way, just that there is at least one way. That's how this definition differs from the definition for basis of a vector space.

An example. The set  $S = \{(1, 3), (2, 2), (3, 1)\}$  spans the vector space  $\mathbf{R}^2$ , but it's not a basis of it. (Why not?)

**The linear combination problem in MATLAB.** Consider the question whether a particular vector  $\mathbf{v}$  is a linear combination of given vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ . This is the same question as: Is  $\mathbf{v}$  in the span of the given vectors?

This question can be solved in MATLAB. After all, you're just looking to solve the vector equation

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_k$$

for the unknowns  $c_1, c_2, \dots, c_n$ , and that's just a system of linear equations.

Here we'll determine if the vector  $\mathbf{v} = (1, 4, 2, 6)$  is a linear combination of the vectors  $\mathbf{x} = (3, -1, 1, 0)$ ,  $\mathbf{y} = (1, 1, 1, 1)$ , and  $\mathbf{z} = (1, 2, -1, 3)$ . Note that the system will have 4 equations (one for each coordinate) in three unknowns (being  $c_1, c_2$ , and  $c_3$ ), so we don't expect it to have a solution. Treat all the vectors as column vectors, place the vectors as columns in an augmented matrix, and row reduce it using the function `rref`. (In fact, I'll enter them as rows, then transpose.)

```
>> aug=[3 -1 1 0;1 1 1 1;1 2 -1 3;1 4 2 6]';
aug =
     3     1     1     1
    -1     1     2     4
     1     1    -1     2
     0     1     3     6

>> rref(aug)
ans =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

Thus the four equations are inconsistent since the last equation says  $0 = 1$ . Thus,  $\mathbf{v}$  is not a linear combination of the others.

**Linear independence.** The question of spanning a vector space asks if you have enough vectors in a set  $S$  to get all other vectors in a space as a linear combination of the vectors in  $S$ . The question of independence asks if you have too many, that is, can you do without some of them because they're redundant.

**Definition 5.** A set

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$$

of vectors in a vector space  $V$  is said to be *linearly dependent* if there are scalars  $c_1, c_2, \dots, c_k$  not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

You can read this as saying that at least one of the vectors is a linear combination of the rest, for if  $c_i \neq 0$ , then  $\mathbf{v}_i$  is a linear combination of the rest.

If the vectors aren't linearly dependent, then we say they're *linearly independent*. In other words, no vector in  $S$  is a linear combination of the others.

A logically equivalent statement is that  $S$  is linearly independent if the only way a linear combination of vectors in  $S$  can equal 0,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

is when each of the scalars  $c_1, c_2, \dots, c_k$  are all 0. In other words,  $\mathbf{0}$  is not a nontrivial linear combination of the vectors in  $S$ .

How do you know whether the vectors in  $S$  are linearly dependent or independent? Just solve the vector equation  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$  for  $c_1, c_2, \dots, c_k$ . This single vector equation is a system of homogeneous linear equations. If you only get the trivial solution, then the vectors in  $S$  are linearly independent. If you get any other solution, then they're dependent.

For  $\mathbf{R}^n$ , the standard unit vectors  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$  are linearly independent. You can see that each of them,  $\mathbf{e}_i$ , is the only one of them with a nonzero  $i^{\text{th}}$  coordinate, therefore it is not a linear combination of the rest. So they're all independent.

In general, two vectors  $\mathbf{v}$  and  $\mathbf{w}$  are linearly independent if and only if each is not a multiple of the other. Geometrically that means they do not lie on the same line through the origin  $\mathbf{0}$ .

**Testing for linear independence using MATLAB.** In order to tell if vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are independent, check to see if the homogeneous system  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$  has any nontrivial solutions. We can do that in MATLAB with the `rref` function.

For example, let's see if the vectors  $(1, 3, 5, 7)$ ,  $(2, 0, 1, 3)$ ,  $(-1, 2, -1, 0)$ , and  $(4, 3, 1, -5)$  are independent. Place them in four columns of a coefficient matrix, and row reduce the matrix.

```
>>A=[1 3 5 7;2 0 1 3;-1 2 -1 0;-5 17 9 15]'
```

```
A =
```

```
 1     2    -1    -5
 3     0     2    17
 5     1    -1     9
 7     3     0    15
```

```
>> rref(A)
```

```
ans =
```

```
 1     0     0     3
 0     1     0    -2
 0     0     1     4
 0     0     0     0
```

There are nontrivial solutions. The unknown  $c_4$  can be chosen freely, and the general solution is  $(c_1, c_2, c_3, c_4) = (-3c_4, 2c_4, -4c_4, c_4)$ . Thus they are not independent. Taking  $c_4 = -1$ , we can write  $\mathbf{0}$  as a nontrivial combination of the four vectors by

$$3\mathbf{v}_1 - 2\mathbf{v}_2 + 4\mathbf{v}_3 - \mathbf{v}_4 = \mathbf{0}.$$

**A basis is a linearly independent spanning set.** We defined basis first, then looked two aspects of a basis, that of span and that of independence. Now, we'll combine the two concepts together jointly mean basis.

**Theorem 6.** A subset  $S$  of a vector space  $V$  is a basis if and only if (1)  $S$  spans  $V$ , and (2)  $S$  is linearly independent.

*Proof.* Part I. Suppose  $S$  is a basis by the definition. Then every vector is a linear combination, so  $S$  spans  $V$ . Also, the vector  $\mathbf{0}$  is uniquely a linear combination of elements of  $S$ , so  $S$  is linearly independent.

Part II. Suppose that  $S$  spans  $V$  and it's linearly independent. Since it spans  $V$ , every vector can be represented as some linear combination of elements

of  $S$ . We have yet to show there's only one such linear combination. Suppose that a vector  $\mathbf{v}$  can be represented in two ways:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

$$\mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_k.$$

Subtracting the second equation from the first yields the equation

$$\mathbf{0} = (c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \cdots + (c_k - d_k)\mathbf{v}_k.$$

But  $S$  is linearly independent, so  $\mathbf{0}$  is only a trivial linear combination of the basis vectors, that is,  $c_i - d_i = 0$  for each index  $i$ . Therefore, each  $c_i = d_i$ . Hence the two representations were the same. Q.E.D.

**Theorem 7.** Given a finite set of vectors spans a vector space, then it has a subset which is a basis for that vector space.

*Proof.* Let the finite set  $S$  span the vector space  $V$ . There are a couple of ways that you can find an independent subset of  $S$  that spans  $V$ .

One way is to throw out redundant vectors in  $S$ . If  $S$  is already independent, you're done. If not, one of the vectors  $\mathbf{v}$  depends on the rest. Then  $S' = S - \{\mathbf{v}\}$  also spans  $V$  since, as  $\mathbf{v}$  is a linear combination of  $S'$ , and every vector is a linear combination of  $\mathbf{v}$  and the others, therefore every vector is a linear combination of just the others. Continue throwing out vectors until you're left with an independent subset that still spans  $V$ . Since there were only a finite number of vectors in  $S$  to begin with, the process will terminate.

The other way is to build up a basis. Go through the vectors in  $S$  one at a time. If the next one is dependent on the previous, then don't include it, otherwise do. When you're all done, you've got an independent subset  $S'$  of  $S$ , and every vector in  $S$  is dependent on it. Since  $S$  spanned  $V$ , so does  $S'$ . Q.E.D.

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