CLARK

Name: ____________________________

Mailbox number:___________________

Math 131 Multivariate Calculus

Final Exam

7 May 2010

You may refer to two sheet of notes on this test. You may use a calculator. You may leave your answers as expressions such as $e^2\sqrt{\dfrac{\sin^2(\pi/6)}{1-\ln 10}}$ if you like. Show all your work for credit. Points for each problem are in square brackets.

1. [16; 8 points each part] On conservative vector fields. We proved that a conservative vector field $\mathbf{F}$ on a simply connected region is the gradient of some scalar field f.

a. Verify that the vector field $\mathbf{F}$ given by $\mathbf{F}(x, y, z) = (2x + y, x + \cos z, -y \sin z)$ has curl 0.

b. Find a scalar potential field f on $\mathbf{R}^3$ whose gradient is $\mathbf{F}$.

2. [16] On Green's theorem. Recall that Green's theorem equates a path integral over the boundary of a two-dimensional region D to a double integral over D.

$$\oint_{\partial D} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$$

Let $\mathbf{F}$ be the vector field defined on $\mathbf{R}^2$ by $\mathbf{F}(x, y) = (y^2, x^2)$. Let C be the path formed by the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, oriented counterclockwise. Use Green's theorem to convert the vector line integral $\oint_C \mathbf{F} \cdot d\mathbf{s}$ into a double integral. Your double integral should have only the variables x and y, and it should have limits of integration for both x and y. Don't evaluate the resulting double integral.

3. [18; 6 points each part] On scalar line integrals. Recall that the scalar line integral of a scalar field f on a path parameterized by $\mathbf{x}$ is

$$\int_{\mathbf{x}} f\, ds = \int_a^b f(\mathbf{x}(t))\|\mathbf{x}'(t)\|\, dt.$$

Tom Sawyer is whitewashing a picket fence. The base of the fenceposts are arranged in the (x, y)-plane as the quarter circle $x^2 + y^2 = 25$ for $x, y \geq 0$, and the height of the fencepost at point (x, y) is given by $h(x, y) = 10 - x - y$. In this problem, you will use a scalar line integral to find the area of one side of the fence.

a. Parameterize the quarter circle by a path $\mathbf{x}(t)$. Be sure to include the limits for the parameter t.

b. Compute the velocity $\mathbf{x}'(t)$ and speed $\|\mathbf{x}'\|$ for your parameterization

c. Write down a scalar line integral of h over the path, and evaluate that integral.

4. [16] On scalar surface integrals. Recall that the integral of a scalar field f over a surface parameterized by $\mathbf{X}$ is

$$\iint_{\mathbf{X}} f\, dS = \iint_D f(\mathbf{X}(s,t))\, \|\mathbf{N}(s,t)\|\, ds\, dt$$

Evaluate the scalar surface integral $\displaystyle\iint_{\mathbf{X}} z^3\, dS$ where $\mathbf{X}$ is the parameterization of the unit hemisphere

$$\mathbf{X}(s,t) = (\cos s \sin t, \sin s \sin t, \cos t), \text{ for } 0 \le s \le 2\pi, 0 \le t \le \pi/2.$$

You may use the fact that the length of the normal vector $\mathbf{N}(s,t)$ is equal to $\sin t$. Carry out your evaluatation until you get an ordinary double integral in terms of s and t. You don't have to evaluate that integral.

5. [20; 5 points each part] On Gauss's theorem. Recall that Gauss's theorem, also known as the divergence theorem, says that the integral of $\mathbf{F}$ over ∂D equals the divergence of $\mathbf{F}$ over the region D.

$$\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

Let D be the segment of a paraboloid $D = \{(x, y, z) \in \mathbf{R}^3 \,|\, 0 \leq z \leq 9 - x^2 - y^2\}$ and let $\mathbf{F}$ be the radial vector field given by $\mathbf{F}(x, y, z) = (x, y, z)$.

a. Write down the triple integral $\iiint_D \nabla \cdot \mathbf{F}\, dV$ in terms of x, y, and z with limits of integration for each. Don't evaluate the integral.

b. The boundary ∂D comes in two parts—S_1, the upper parabolic surface, and S_2, the lower surface which is a circle of radius 3 in the x, y-plane. Parameterize the surface S_1.

c. Compute the normal vector **N** for the parameterization you chose in part **b**. You'll use **N** in part **d.**

d. Recall that the vector surface integral of a vector field **F** on a surface parameterized by **X** is

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t)\, ds\, dt.$$

Write down the surface integral $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$ for the upper parabolic surface in terms of the two variables you used in your parameterization of S_1 with limits of integration for those two variables. No other variables should appear in your final integral. Don't evaluate the integral.

6. [16] On change of variables and the Jacobian.

Parabolic coordinates. The relevant equations to convert between rectangular coordinates (x, y) and parabolic coordinates (u, v) are

$$x = uv \qquad u = \sqrt{\sqrt{x^2 + y^2} + y}$$
$$y = \tfrac{1}{2}(u^2 - v^2) \qquad v = \sqrt{\sqrt{x^2 + y^2} - y}$$

A double integral can be converted from rectangular coordinates to parabolic coordinates using a Jacobian. The area differential $dA = dx\,dy$ is equal to $\left|\dfrac{\partial(x, y)}{\partial(u, v)}\right| du\,dv$.

Determine the Jacobian

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| =$$

#1.[16]	
#2.[16]	
#3.[18]	
#4.[16]	
#5.[20]	
#6.[16]	
Total	