Math 131 Multivariate Calculus
Final Answers
May 2010

Scale. ...  

1. [16; 8 points each part] On conservative vector fields. We proved that a conservative vector field \( \mathbf{F} \) on a simply connected region is the gradient of some scalar field \( f \).

**a.** Verify that the vector field \( \mathbf{F} \) given by \( \mathbf{F}(x, y, z) = (2x + y, x + \cos z, -y \sin z) \) has curl 0.

\[
\text{curl} \ \mathbf{F} = \nabla \times \mathbf{F} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (F_1, F_2, F_3)
\]

\[
= \left| \begin{array}{ccc}
i & j & k \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1 & F_2 & F_3
\end{array} \right|
\]

\[
= \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)
\]

\[
= (-\sin z + \sin z, 0 - 0, 1 - 1) = (0, 0, 0)
\]

**b.** Find a scalar potential field \( f \) on \( \mathbb{R}^3 \) whose gradient is \( \mathbf{F} \).

Since \( \frac{\partial f}{\partial x} = 2x + y \), therefore \( f(x, y, z) = x^2 + xy + C(y, z) \) where \( C(y, z) \) can depend on \( y \) and \( z \) but not on \( x \). Take \( \frac{\partial}{\partial y} \) to see that we need \( x + \frac{\partial}{\partial y} C(y, z) = x + \cos z \). Therefore, \( C(y, z) = y \cos z + \text{some function of } z \). The function

\[
f(x, y, z) = x^2 + xy + y \cos z
\]

will do since its derivative with respect to \( z \) is \(-y \sin z\) as required.

2. [16] On Green’s theorem. Recall that Green’s theorem equates a path integral over the boundary of a two-dimensional region \( D \) to a double integral over \( D \).

\[
\oint_{\partial D} M \, dx + N \, dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy.
\]

Let \( \mathbf{F} \) be the vector field defined on \( \mathbb{R}^2 \) by \( \mathbf{F}(x, y) = (y^2, x^2) \). Let \( C \) be the path formed by the square with vertices \((0, 0), (1, 0), (1, 1), \) and \((0, 1)\), oriented counterclockwise. Use Green’s theorem to convert the vector line integral

\[
\int_C \mathbf{F} \cdot ds
\]

into a double integral. Your double integral should have only the variables \( x \) and \( y \), and it should have limits of integration for both \( x \) and \( y \). Don’t evaluate the resulting double integral.

The closed curve \( C \) is the boundary \( \partial D \) of the unit square \( D \), so by Green’s theorem, the vector line integral is equal to

\[
\int_D \left( \frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} y^2 \right) \, dx \, dy = \int_0^1 \int_0^1 (2x - 2y) \, dx \, dy.
\]

3. [18; 6 points each part] On scalar line integrals. Recall that the scalar line integral of a scalar field \( f \) on a path parameterized by \( x \) is

\[
\int_a^b f(x(t)) \|x'(t)\| \, dt.
\]

Tom Sawyer is whitewashing a picket fence. The base of the fenceposts are arranged in the \((x, y)\)-plane as the quarter circle \( x^2 + y^2 = 25 \) for \( x, y \geq 0 \), and the height of the fencepost at point \((x, y)\) is given by \( h(x, y) = 10 - x - y \). In this problem, you will use a scalar line integral to find the area of one side of the fence.

**a.** Parameterize the quarter circle by a path \( x(t) \). Be sure to include the limits for the parameter \( t \).

\[
x(t) = (5 \cos t, 5 \sin t) \quad 0 \leq t < \pi/2.
\]

**b.** Compute the velocity \( x'(t) \) and speed \( \|x'(t)\| \) for your parameterization.

For this path, the velocity is \( x'(t) = (-5 \sin t, 5 \cos t) \), so the speed is \( \|x'(t)\| = 5 \).

**c.** Write down a scalar line integral of \( h \) over the path, and evaluate that integral.

\[
\int_x (10 - x - y) \, ds = \int_0^{\pi/2} (10 - x) \|x'(t)\| \, dt
\]

\[
= 5 \int_0^{\pi/2} (10 - 5 \cos t - 5 \sin t) \, dt
\]

\[
= 25(\pi - 2)
\]

4. [16] On scalar surface integrals. Recall that the integral of a scalar field \( f \) over a surface parameterized by \( \mathbf{X} \) is

\[
\iint_X f \, dS = \iint_D f(\mathbf{X}(s, t)) \|\mathbf{N}(s, t)\| \, ds \, dt.
\]

Evaluate the scalar surface integral \( \iint_X z^3 \, dS \) where \( \mathbf{X} \) is the parameterization of the unit hemisphere \( \mathbf{X}(s, t) = (\cos s \sin t, \sin s \sin t, \cos t) \) for \( 0 \leq s \leq 2\pi \) and \( 0 \leq t \leq \pi/2 \). You may use the fact that the length of the normal vector \( \mathbf{N}(s, t) \) is equal to \( \sin t \). Carry out your evaluation until you get an ordinary double integral in terms of \( s \) and \( t \). You don’t have to evaluate that integral.

\[
\iint_X z^3 \, dS = \iint_D \cos^3 t \sin t \, ds \, dt
\]

\[
= \int_0^{\pi/2} \int_0^{2\pi} \cos^3 t \sin t \, ds \, dt
\]
5. [20; 5 points each part] On Gauss’s theorem. Recall that Gauss’s theorem, also known as the divergence theorem, says that the integral of $\mathbf{F}$ over $\partial D$ equals the divergence of $\mathbf{F}$ over the region $D$.

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \int \int_D \nabla \cdot \mathbf{F} \, dV$$

Let $D$ be the segment of a paraboloid $D = \{(x, y, z) \in \mathbb{R}^3 | 0 \leq z \leq 9 - x^2 - y^2\}$ and let $\mathbf{F}$ be the radial vector field given by $\mathbf{F}(x, y, z) = (x, y, z)$.

a. Write down the triple integral $\int \int \int_D \nabla \cdot \mathbf{F} \, dV$ in terms of the two variables you used in your parameterization of $S_1$ with limits of integration for those two variables. No other variables should appear in your final integral. Don’t evaluate the integral.

$$\int \int \int_{S_1} \mathbf{F} \cdot \mathbf{N} \, ds \, dt$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-s^2}}^{\sqrt{9-s^2}} \int_{0}^{9-x^2-y^2} 3 \, dz \, dy \, dx$$

b. The boundary $\partial D$ comes in two parts—$S_1$, the upper parabolic surface, and $S_2$, the lower surface which is a circle of radius 3 in the $x, y$-plane. Parameterize the surface $S_1$.

There are various ways to do that. Here’s one. Take $x$ and $y$ to be the parameters. You could leave them as $x$ and $y$, but I’ll write them as $s$ and $t$ for clarity. Then $x = s, y = t$ and $z = 9 - s^2 - t^2$, where $-3 \leq s \leq 3$ and $-\sqrt{9-s^2} \leq t \leq \sqrt{9-s^2}$.

c. Compute the normal vector $\mathbf{N}$ for the parameterization you chose in part b. You’ll use $\mathbf{N}$ in part d.

There are various ways to compute $\mathbf{N}$. You could use the formula $\mathbf{N} = (-f_x, -f_y, 1)$ that we developed in class for the normal for the graph of a function $z = f(x, y) = f(s, t)$.

Here’s a way to compute $\mathbf{N}$ that uses Jacobians. It leads to the formula mentioned above.

$$\mathbf{N}(s, t) = \left( \frac{\partial (y, z)}{\partial (s, t)}, \frac{\partial (x, z)}{\partial (s, t)}, \frac{\partial (x, y)}{\partial (s, t)} \right)$$

$$= \left( \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} - \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} - \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} - \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \right)$$

$$= \left( \frac{\partial z}{\partial x}, \frac{\partial z}{\partial t}, 1 \right)$$

$$= (2x, 2y, 1)$$

d. Recall that the vector surface integral of a vector field $\mathbf{F}$ on a surface parameterized by $\mathbf{X}$ is

$$\int \int_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \int \int_{D} \mathbf{F}(\mathbf{X}(s, t)) \cdot \mathbf{N}(s, t) \, ds \, dt.$$