Math 131 Multivariate Calculus
Final Answers
May 2010

Scale. ...

1. [16; 8 points each part] On conservative vector fields. We proved that a conservative vector field $\mathbf{F}$ on a simply connected region is the gradient of some scalar field f.

a. Verify that the vector field $\mathbf{F}$ given by $\mathbf{F}(x, y, z) = (2x + y, x + \cos z, -y \sin z)$ has curl 0.

$$\begin{aligned}
\text{curl } \mathbf{F} &= \nabla \times \mathbf{F} \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_1, F_2, F_3) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\
&= (-\sin z + \sin z, 0 - 0, 1 - 1) = (0, 0, 0)
\end{aligned}$$

b. Find a scalar potential field f on $\mathbf{R}^3$ whose gradient is $\mathbf{F}$.

Since $\frac{\partial f}{\partial x} = 2x + y$, therefore $f(x, y, z) = x^2 + xy + C(y, z)$ where $C(y, z)$ can depend on y and z but not on x. Take $\frac{\partial}{\partial y}$ to see that we need $x + \frac{\partial}{\partial y} C(y, z) = x + \cos z$. Therefore, $C(y, z) = y \cos z$ plus some function of z. The function

$$f(x, y, z) = x^2 + xy + y \cos z$$

will do since its derivative with respect to z is $-y \sin z$ as required.

2. [16] On Green's theorem. Recall that Green's theorem equates a path integral over the boundary of a two-dimensional region D to a double integral over D.

$$\oint_{\partial D} M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Let $\mathbf{F}$ be the vector field defined on $\mathbf{R}^2$ by $\mathbf{F}(x, y) = (y^2, x^2)$. Let C be the path formed by the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$, oriented counterclockwise. Use Green's theorem to convert the vector line integral $\oint_C \mathbf{F} \cdot d\mathbf{s}$ into a double integral. Your double integral should have only the variables x and y, and it should have limits of integration for both x and y. Don't evaluate the resulting double integral.

The closed curve C is the boundary ∂D of the unit square D, so by Green's theorem, the vector line integral is equal to

$$\int_D \left(\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} y^2\right) dx\,dy = \int_0^1 \int_0^1 (2x - 2y)\,dx\,dy.$$

3. [18; 6 points each part] On scalar line integrals. Recall that the scalar line integral of a scalar field f on a path parameterized by $\mathbf{x}$ is

$$\int_{\mathbf{x}} f\,ds = \int_a^b f(\mathbf{x}(t)) \|\mathbf{x}'(t)\|\,dt.$$

Tom Sawyer is whitewashing a picket fence. The base of the fenceposts are arranged in the (x, y)-plane as the quarter circle $x^2 + y^2 = 25$ for $x, y \geq 0$, and the height of the fencepost at point (x, y) is given by $h(x, y) = 10 - x - y$. In this problem, you will use a scalar line integral to find the area of one side of the fence.

a. Parameterize the quarter circle by a path $\mathbf{x}(t)$. Be sure to include the limits for the parameter t.

$\mathbf{x}(t) = (5 \cos t, 5 \sin t)$ for $0 \leq t \leq \pi/2$.

b. Compute the velocity $\mathbf{x}'(t)$ and speed $\|\mathbf{x}'\|$ for your parameterization.

For this path, the velocity is $\mathbf{x}'(t) = (-5 \sin t, 5 \cos t)$, so the speed is $\|\mathbf{x}'(t)\| = 5$.

c. Write down a scalar line integral of h over the path, and evaluate that integral.

$$\begin{aligned}
\int_{\mathbf{x}} (10 - x - y)\,ds &= \int_0^{\pi/2} (10 - x - y)\,\|\mathbf{x}'(t)\|\,dt \\
&= 5 \int_0^{\pi/2} (10 - 5\cos t - 5 \sin t)\,dt \\
&= 25(\pi - 2)
\end{aligned}$$

4. [16] On scalar surface integrals. Recall that the integral of a scalar field f over a surface parameterized by $\mathbf{X}$ is

$$\iint_{\mathbf{X}} f\,dS = \iint_D f(\mathbf{X}(s, t))\,\|\mathbf{N}(s, t)\|\,ds\,dt$$

Evaluate the scalar surface integral $\iint_{\mathbf{X}} z^3\,dS$ where $\mathbf{X}$ is the parameterization of the unit hemisphere $\mathbf{X}(s, t) = (\cos s \sin t, \sin s \sin t, \cos t)$ for $0 \leq s \leq 2\pi$ and $0 \leq t \leq \pi/2$. You may use the fact that the length of the normal vector $\mathbf{N}(s, t)$ is equal to $\sin t$. Carry out your evaluatation until you get an ordinary double integral in terms of s and t. You don't have to evaluate that integral.

$$\begin{aligned}
\iint_{\mathbf{X}} z^3\,dS &= \iint_D \cos^3 t\,|\sin t|\,ds\,dt \\
&= \int_0^{\pi/2} \int_0^{2\pi} \cos^3 t\,\sin t\,ds\,dt
\end{aligned}$$

5. [20; 5 points each part] On Gauss's theorem. Recall that Gauss's theorem, also known as the divergence theorem, says that the integral of $\mathbf{F}$ over ∂D equals the divergence of $\mathbf{F}$ over the region D.

$$\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

Let D be the segment of a paraboloid $D = \{(x, y, z) \in \mathbf{R}^3 \,|\, 0 \le z \le 9 - x^2 - y^2\}$ and let $\mathbf{F}$ be the radial vector field given by $\mathbf{F}(x, y, z) = (x, y, z)$.

a. Write down the triple integral $\iiint_D \nabla \cdot \mathbf{F}\, dV$ in terms of x, y, and z with limits of integration for each. Don't evaluate the integral.

The divergence of $\mathbf{F}$ is $\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3$. One way to paramterize the integral is

$$\iiint_D 3\, dV = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} 3\, dz\, dy\, dx$$

b. The boundary ∂D comes in two parts—S_1, the upper parabolic surface, and S_2, the lower surface which is a circle of radius 3 in the x, y-plane. Parameterize the surface S_1.

There are various ways to do that. Here's one. Take x and y to be the parameters. You could leave them as x and y, but I'll write them as s and t for clarity. Then $x = s$, $y = t$ and $z = 9 - s^2 - t^2$, where $-3 \le s \le 3$ and $-\sqrt{9 - s^2} \le t \le \sqrt{9 - s^2}$

c. Compute the normal vector $\mathbf{N}$ for the parameterization you chose in part **b**. You'll use $\mathbf{N}$ in part **d.**

There are various ways to compute $\mathbf{N}$. You could use the formula $\mathbf{N} = (-f_x, -f_t, 1)$ that we developed in class for the normal for the graph of a function $z = f(x, y) = f(s, t)$.

Here's a way to compute $\mathbf{N}$ that uses Jacobians. It leads to the formula mentioned above.

$$\begin{aligned}
\mathbf{N}(s,t) &= \left(\frac{\partial(y,z)}{\partial(s,t)}, \frac{\partial(x,z)}{\partial(s,t)}, \frac{\partial(x,y)}{\partial(s,t)} \right) \\
&= \left(\frac{\partial y}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial z}{\partial s}, \frac{\partial x}{\partial s}\frac{\partial z}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial z}{\partial s}, \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \right) \\
&= \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial t}, 1 \right) \\
&= (2x, 2y, 1)
\end{aligned}$$

d. Recall that the vector surface integral of a vector field $\mathbf{F}$ on a surface parameterized by $\mathbf{X}$ is

$$\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t)\, ds\, dt.$$

Write down the surface integral $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S}$ for the upper parabolic surface in terms of the two variables you used in your parameterization of S_1 with limits of integration for those two variables. No other variables should appear in your final integral. Don't evaluate the integral.

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_{S_1} \mathbf{F} \cdot \mathbf{N}\, ds\, dt \\
&= \int_{-3}^{3} \int_{-\sqrt{9-s^2}}^{\sqrt{9-s^2}} (x, y, z) \cdot (2x, 2y, 1)\, dt\, ds \\
&= \int_{-3}^{3} \int_{-\sqrt{9-s^2}}^{\sqrt{9-s^2}} (2x^2 + 2y^2 + z)\, dt\, ds \\
&= \int_{-3}^{3} \int_{-\sqrt{9-s^2}}^{\sqrt{9-s^2}} (2s^2 + 2t^2 + 9 - s^2 - t^2)\, dt\, ds \\
&= \int_{-3}^{3} \int_{-\sqrt{9-s^2}}^{\sqrt{9-s^2}} (9 + s^2 + t^2)\, dt\, ds
\end{aligned}$$

6. [16] On change of variables and the Jacobian.

Parabolic coordinates. The relevant equations to convert between rectangular coordinates (x, y) and parabolic coordinates (u, v) are

$$\begin{aligned}
x &= uv & u &= \sqrt{\sqrt{x^2 + y^2} + y} \\
y &= \tfrac{1}{2}(u^2 - v^2) & v &= \sqrt{\sqrt{x^2 + y^2} - y}
\end{aligned}$$

A double integral can be converted from rectangular coordinates to parabolic coordinates using a Jacobian. The area differential $dA = dx\, dy$ is equal to $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$.

Determine the Jacobian $\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$.

$$\begin{aligned}
\frac{\partial(x,y)}{\partial(u,v)} &= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \\
&= -v^2 - u^2
\end{aligned}$$

The area differential includes an absolute value, and that's $v^2 + u^2$.