# Math 131 Multivariate Calculus

First Test Answers

Feb. 2010

**Scale.** 90–101 A, 80–89 B, 65–79 C. Median 95.

**Problem 1.** [18; 6 points each part] On functions of several variables.

**a.** Give an example of a function $\mathbf{f} : \mathbf{R} \to \mathbf{R}^3$ and another example $g : \mathbf{R}^3 \to \mathbf{R}$.

Typical sample functions are

$$\mathbf{f}(t) = (\cos t, \sin t, t)$$

and

$$g(x, y, z) = x^2 + y^2 + z^2.$$

**b.** Give an example of a vector-valued function $\mathbf{f}$ whose domain is the set

$$\{(x, y) \in \mathbf{R}^2 | x > 0 \text{ and } y > 0\}.$$

A typical one is $\mathbf{x}(x, y) = (\ln x, \ln y)$.

**c.** Explain why all the level curves for $f(x, y) = x^2 + y^2$ at positive heights $c$ are circles.

The level curve for $f$ at height $c$ has the equation $x^2 + y^2 = c$, which is the equation of a circle of radius $\sqrt{c}$ in the plane $z = c$.

**Problem 2.** [20; 10 points each part] On limits and continuity.

**a.** Explain why the limit, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$, does not exist.

One explanation: as $(x, y) \to (0, 0)$ along either the $x$-axis or the $y$-axis, the expression is constantly 0. But along some other line, say $y = x$,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}.$$

Since the limiting value depends on the approach, the limit doesn't exist.

**b.** Explain why the function $f(x, y) = \sin(3x + 2y)$ is contінous throughout its domain.

It's the composition of two continuous functions, namely, the sine function and a polynomial.

**Problem 3.** [32; 8 points each part] On derivatives.

**a.** Compute the gradient $\nabla f$ if $f(x, y, z) = e^{x+yz}$.

$$\begin{aligned}\nabla f(x, y, z) &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \\ &= \left(e^{x+yz}, ze^{x+yz}, ye^{x+yz}\right)\end{aligned}$$

**b.** Find $\dfrac{\partial^2 f}{\partial x \partial y}$ for the function $f$ given in part **a**.

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} ze^{x+yz} = ze^{x+yz}$$

**c.** For the function $f$ in part **a** determine the directional derivative in the direction $\mathbf{u} = (0, \frac{3}{5}, \frac{4}{5})$.

$$\nabla f(x, y, z) \cdot \mathbf{u} = 0e^{x+yz} + \frac{3}{5} ze^{x+yz} + \frac{4}{5} ye^{x+yz}$$

**d.** Find the derivative $D\mathbf{f}$ if

$$\mathbf{f}(x, y) = (x^3 + 3x^2y + 3xy^2 + y^3, \sin x + \cos y, x/y).$$

Since $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^3$, therefore $D\mathbf{f}$ is a $3 \times 2$ matrix.

$$D\mathbf{f} = \begin{bmatrix} 3x^2 + 6xy + 3y^2 & 3x^2 + 6xy + 3y^2 \\ \cos x & -\sin y \\ 1/y & -x/y^2 \end{bmatrix}$$

**Problem 4.** [15] On the chain rule. Suppose that $f : \mathbf{R}^3 \to \mathbf{R}^2$ has the derivative

$$D\mathbf{f}(x, y) = \begin{bmatrix} \sin y & x \cos y & 0 \\ 2x & 2y & 2z \end{bmatrix}$$

and $\mathbf{x} : \mathbf{R}^2 \to \mathbf{R}^3$ has the derivative

$$D\mathbf{f}(s, t) = \begin{bmatrix} 2s & 0 \\ 2t & 2s \\ 0 & 2t \end{bmatrix}.$$

**a.** [5] The derivative $D(\mathbf{f} \circ \mathbf{x})(s, t)$ is a matrix. What size is that matrix?

Since $\mathbf{f} \circ \mathbf{x} : \mathbf{R}^2 \to \mathbf{R}^2$, it's a $2 \times 2$ matrix.

**b.** [10] Find the derivative $D(\mathbf{f} \circ \mathbf{x})(s, t)$. (You may leave your answer in terms of the variables $x$, $y$, $z$, $s$, and $t$.)

$$D(\mathbf{f} \circ \mathbf{x})(s, t) = \begin{bmatrix} 2s \sin y + 2tx \cos y & 2sx \cos y \\ 4sx + 4ty & 4sy + 4tz \end{bmatrix}$$

**Problem 5.** [16; 4 points each part] On paths.

**a.** Give an example of a path $\mathbf{x} : \mathbf{R} \to \mathbf{R}^2$ that passes through the point $(2, 4) \in \mathbf{R}^2$.

For example, $\mathbf{x}(t) = (x(t), y(t)) = (t, t^2)$. The path goes through $(2, 4)$ when $t = 2$.

**b.** What is its velocity as it passes through $(2, 4)$?

For the example, $\mathbf{x}'(t) = (1, 2t)$. So at $t = 2$, $\mathbf{x}'(2) = (2, 4)$.

**c.** What is its speed as it passes through $(2, 4)$?

$\|\mathbf{x}'(2)\| = \sqrt{2^2 + 4^2} = \sqrt{20}$.

**d.** What is its acceleration as it passes through $(2, 4)$?

$\mathbf{x}''(t) = (x''(t), y''(t)) = (0, 2)$.