Math 131 Multivariate Calculus

Second Test Answers
April 2010

Scale. 90–100 A, 77–89 B, 61–76 C. Median 89.

1. [20; 10 points each part] Consider the path in $\mathbf{R}^2$ given by $\mathbf{x}(t) = (t^2, \frac{2}{3}(2t+1)^{3/2})$ for $0 \le t \le 4$.

a. Write down an integral which gives the length of that path.

In general, the length is $\int_a^b \|\mathbf{x}'(t)\|\,dt$ where

$$\|\mathbf{x}'(t)\| = \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2}.$$

For this function $\frac{\partial x}{\partial t} = 2t$ while $\frac{\partial y}{\partial t} = 2(2t+1)^{1/2}$. Therefore, the length is given by the integral

$$\int_0^4 \sqrt{4t^2 + 4(2t+1)}\,dt = \int_0^4 2(t+1)\,dt.$$

b. Evaluate that integral.

It's $t^2 + 2t\Big|_0^4 = 24$.

2. [15] Show that the vector field

$$\mathbf{F}(x,y,z) = (10x + 2xz^2, 28y^3, 2x^2z)$$

is irrotational.

A vector field is irrotational if its curl is $\mathbf{0}$. The curl of this vector field is

$$\begin{aligned}
\text{curl}\,\mathbf{F} &= \nabla \times \mathbf{F} \\
&= \left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}\right) \times (F_1, F_2, F_3) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \left(\tfrac{\partial F_3}{\partial y} - \tfrac{\partial F_2}{\partial z}, \tfrac{\partial F_1}{\partial z} - \tfrac{\partial F_3}{\partial x}, \tfrac{\partial F_2}{\partial x} - \tfrac{\partial F_1}{\partial y}\right) \\
&= (0-0, 4xz - 4xz, 0-0) = \mathbf{0}
\end{aligned}$$

3. [15] Calculate the Hessian matrix $Hf(\mathbf{a})$ for the scalar field $f(x,y,z) = x^3 + x^2y - yz^2 + 2z^4$ at the point $\mathbf{a} = (1,0,1)$.

The first and second derivatives of f are

$$f_x = 3x^2 + 2xy, f_y = x^2 - z^2, f_z = -2yz + 8z^3$$

$$f_{xx} = 6x, f_{xy} = 2x, f_{xz} = 0, f_{yy} = 0$$
$$f_{yz} = -2z, f_{zz} = -2yz + 24z^2$$

So the Hessian is

$$\begin{aligned}
Hf(\mathbf{a}) &= \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}(\mathbf{a}) \\
&= \begin{bmatrix} 6 & 2 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 24 \end{bmatrix}
\end{aligned}$$

4. [12] Give an example function $f : \mathbf{R}^2 \to \mathbf{R}$ which has a saddle point at $\mathbf{a} = (1,1)$.

Recall that a critical point is a saddle point it's neither a max nor a min, but nearby the function takes both larger and smaller values than at the critical point.

We looked at the function $f(x,y) = x^2 - y^2$ as a paradigm for saddle points. Its saddle point was at $(0,0)$. Probably this simplest such function with a saddle point at $(1,1)$ can by translating that function to get $f(x,y) = (x-1)^2 - (y-1)^2$.

5. [15] Use Lagrange multipliers to identify the critical points of the function $f(x,y) = 5x + 2y$ subject to the constraint $5x^2 + 2y^2 = 14$.

Besides the constraint, the other two equations come from $\nabla f = \lambda \nabla g$ where $g(x,y) = 5x^2 + 2y^2$. Here, $\nabla f(x,y) = (5,2)$ and $\nabla g(x,y) = (10x, 4y)$, so those are the equations

$$\begin{aligned}
5 &= \lambda 10x \\
2 &= \lambda 4y
\end{aligned}$$

Thus $\lambda = \frac{1}{2}x = \frac{1}{2}y$. Hence, $x = y$. Along with the constraint $5x^2 + 2y^2 = 14$, that determines the two critical points, $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$.

6. [12] Evaluate the double integral $\int_0^2 \int_0^{x^2} (x-y)\,dy\,dx$.

$$\begin{aligned}
&= \int_0^2 \left(xy - \tfrac{1}{2}y^2\right)\Big|_0^{x^2} dx \\
&= \int_0^2 \left(x^3 - \tfrac{1}{2}x^4\right) dx \\
&= \tfrac{1}{4}x^4 - \tfrac{1}{10}x^5\Big|_0^2 = \tfrac{16}{4} - \tfrac{32}{10} = \tfrac{4}{5}
\end{aligned}$$

7. [12] Set up a double integral to compute the volume of a solid whose base is the plane region D bounded by $x = 2$, $x = 5$, and $x + y = 2$, and $y = x^2$; and whose height at a point (x,y) in that region is given by $f(x,y) = ye^x$. Do not evaluate the integral.

When $2 \le x \le 5$, $x^2 > 2 - y$, so the double integral is

$$\int_2^5 \int_{2-x}^{x^2} ye^x\,dy\,dx.$$