Math 131 Multivariate Calculus
First Test Answers
Feb 2014

Scale. 89–102 A, 76–88 B, 52–75 C. Median 81.

1. [18; 6 points each part] On functions.

a. Find the domain of the function

$$g(x,y) = \frac{1}{\sqrt{4 - x^2 - y^2}}.$$

In order for $g(x,y)$ to be defined, the denominator can't be 0 and $4 - x^2 - y^2$ can't be negative. That means that $x^2 + y^2 < 4$ is required. Thus, the domain is the interior of the circle of radius 2 centered at the origin.

b. Find the range of the function g of part **a**.

Since the denominator can be any number in the interval $(0, 2]$, therefore its reciprocal can be any number in the interval $[\frac{1}{2}, \infty)$, so that interval is the range of g.

c. Consider the function $z = f(x,y) = y - x^2$. Draw either the level curve or the contour curve for f at height $c = 2$. (Draw one or the other. It's your choice.)

The equation $y - x^2 = 2$ is a parabola opening upward with its vertex on the y-axis at $y = 2$. This parabola in the (x,y)-plane is the level curve at height $c = 2$. The corresponding contour curve is the parabola $\{(x,y,2) \,|\, y - x^2 = 2\}$ which lies in the plane $z = 2$.

2. [18; 9 points each part] Evaluate the limit, or explain why it fails to exist.

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

Along the x-axis, y is 0, so the limit along it is 1; but along the y-axis, x is 0, so the limit along it is 0. Since the limit depends on the direction that (x,y) approaches $(0,0)$, therefore the limit doesn't exist.

b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + xy^2 + 2x^2 + 2y^2}{x^2+y^2}$

The rational function simplifies to $x + 2$, so the limit is 2.

3. [18; 9 points each part] On derivatives.

a. Let $f(x,y) = e^{x^2+y^2}$. Compute the partial derivative $\dfrac{\partial f}{\partial x}$.

It's $\dfrac{\partial f}{\partial x} = 2xe^{x^2+y^2}$.

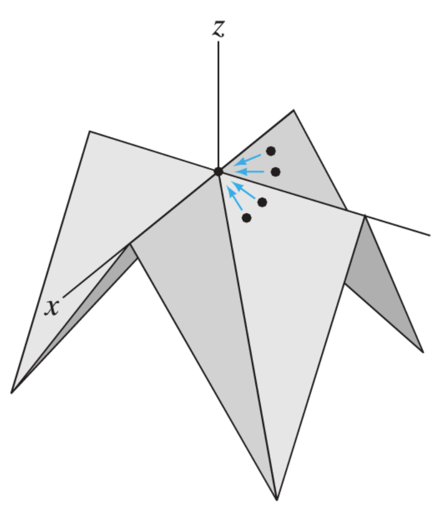
b. Give an example of a function $g : \mathbf{R}^2 \to \mathbf{R}$ whose gradient is $\nabla g = (2xy, x^2)$.

$g(x,y) = x^2y$ will do. The general solution is $g(x,y) = x^2y + C$ where C is any constant.

4. [15] On differentiability. Consider the function whose graph $z = f(x,y)$ is displayed below. It is defined in terms of absolute values by

$$f(x,y) = |(|x| - |y|)| - |x| - |y|.$$

As you can see from the graph, it's partial derivatives evaluated at the origin are both 0, that is, $f_x(0,0) = f_y(0,0) = 0$. Explain why this function is not differentiable at the origin even though it's partial derivatives are both 0.



The surface has no tangent plane at the origin. There are problems in every direction except in the directions of the two axes. For instance, along the line $y = x$, you see valleys on either side of the the origin. On the negative half of this line, the slope is a positive constant, but on the positive half, it's a negative constant. In other words, the intersection of this surface with the plane $x = y$ is a curve with a corner at the origin.

Note that this function is continuous everywhere. It has derivatives everywhere except where the surface $z = f(x, y)$ has a crease.

5. [15] On the chain rule. On the chain rule. Suppose that $f : \mathbf{R}^2 \to \mathbf{R}^3$ has the derivative

$$D\mathbf{f}(x, y) = \begin{bmatrix} 2xy & x^2 \\ 1 & 2y \\ y\cos xy & x\cos xy \end{bmatrix}$$

and $\mathbf{x} : \mathbf{R} \to \mathbf{R}^2$ has the derivative $\mathbf{x}' = (3t^2, t)$.

The composition $\mathbf{y} = \mathbf{f} \circ \mathbf{x}$ is a function $\mathbf{R} \to \mathbf{R}^3$. Find its derivative $\mathbf{y}'$. (Leave your answer in terms of the variables x, y, and t.)

$$\begin{aligned} \mathbf{y}' = D\mathbf{F} \circ \mathbf{x}' &= \begin{bmatrix} 2xy & x^2 \\ 1 & 2y \\ y\cos xy & x\cos xy \end{bmatrix} \begin{bmatrix} 3t^2 \\ t \end{bmatrix} \\ &= \begin{bmatrix} cc6t^2xy + tx^2 \\ 3t^2 + 2ty \\ 3t^2 y\cos xy + tx\cos xy \end{bmatrix} \end{aligned}$$

Problem 6. [18; 9 points each part] On derivatives and directions. Let $f(x, y, z) = x(y^2 + z)$ Note that the gradient of f is

$$\nabla f(x, y, z) = (y^2 + z, 2xy, x).$$

a. In which direction does f increase fastest at the point $\mathbf{a} = (1, 2, 3)$. (For your answer, it's enough to give a vector in that direction. It doesn't have to be a unit vector.)

It increases fastest in the direction of the gradient, $\nabla f(\mathbf{a}) = (7, 4, 1)$

b. At the point $\mathbf{a} = (1, 2, 3)$, evaluate the directional derivative in the direction $\mathbf{u} = (\frac{4}{9}, \frac{7}{9}, \frac{4}{9})$.

The directional derivative is $\nabla f(\mathbf{a}) \cdot \mathbf{u}$, and that equals $(\frac{4}{9}, \frac{7}{9}, \frac{4}{9}) \cdot (7, 4, 1) = \frac{60}{9} = \frac{20}{3}$.