CLARK UNIVERSITY 1887

The arclength parameter s
Math 131 Multivariate Calculus

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Remark on notation. Throughout this discussion we'll be considering a moving point, that is, a path $\mathbf{x} : [a, b] \to \mathbf{R}^n$. We'll take t to be the independent variable, which we'll call time, and we'll use the prime notation to always mean the derivative with respect to t, so, for instance, $\mathbf{x}' = \dfrac{d\mathbf{x}}{dt}$. Whenever we need the derivative with respect to another variable, such as s, we'll stick to Leibniz' notation $\dfrac{d}{ds}$.

Arclength. Consider a path $\mathbf{x} : [a, b] \to \mathbf{R}^n$. We want to define the length L of this path, also called its *arclength*. It will be defined as the integral of its speed. In order for the speed $\|\mathbf{x}'(t)\|$ to have an integral, we'll assume that $\mathbf{x}$ is C^1 (which means it is differentiable and its derivative is continuous).

An argument by analogy for this definition is as follows. In the one-dimensional case, the derivative of position $x(t)$ is velocity $v(t) = x'(t)$, and, by the fundamental theorem of calculus, the integral $\displaystyle\int_a^b v(t)\,dt$ of velocity is the difference in position,

$$\int_a^b v(t)\,dt = x(b) - x(a).$$

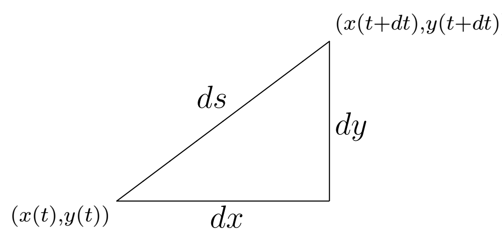
When the velocity is positive, the integral is the distance travelled over the time interval $[a, b]$, that is, the length of the path along the x-axis. But when the velocity is negative, the integral gives the negation of the distance travelled. By replacing velocity by its absolute value, that is by speed $|x'(t)|$, we get the total distance travelled. Furthermore, this integral of speed,

$$\int_a^b |x'(t)|\,dt$$

always works to give the total distance L travelled along the x-axis even when the object alternates the direction it moves along the x-axis.

By analogy, you would guess that even when the direction of travel is not restricted to the x-axis, the distance L travelled along the curve should still be the integral of the speed, and that works out to be right.

A better argument than by analogy uses infinitesimals. Imagine the time interval $[a, b]$ to be divided into infinitesimally short intervals $[t, t+dt]$. The dt is called the *differential* of time t. During this infinitesimally short interval, the object moves from position $\mathbf{x}(t)$ to $\mathbf{x}(t + dt)$. Let's assume we're in dimension 2, so that $\mathbf{x}(t) = (x(t), y(t))$. Then the x-coodinate changes from $x(t)$ to $x(t+dt)$, a difference we can denote $dx = x(t+dt) - x(t)$, called the differential of x. Likewise, in the y-coordinate, we get $dy = y(t+dt) - y(t)$. Those are two sides of an infinitely small right triangle with legs dx and dy, called *Leibniz' differential triangle*.



Let ds denote the hypotenuse of this infinitesimal triangle, so that

$$ds^2 = dx^2 + dy^2$$

and

$$\boxed{ds = \sqrt{dx^2 + dy^2}}$$

Now, during the interval $[t, t + dt]$ the object moves a distance ds. If we sum all these infinitesimal distances, we should get the total distance travelled, that is, the length of the path is the integral

$$\boxed{L = \int_a^b ds = \int_a^b \frac{ds}{dt}\,dt}$$

We can find other expressions for $\frac{ds}{dt}$ as follows.

$$\frac{ds}{dt} = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} = \sqrt{(x')^2 + (y')^2} = \|\mathbf{x}'\|.$$

Thus, the length of the path is the integral of the speed

$$\boxed{L = \int_a^b \|\mathbf{x}'(t)\|\, dt}$$

This formula works in all dimensions, not just $n = 2$.

This argument by infinitesimals is the kind that Leibniz used since he based his calculus on differentials, but arguments by infinitesimals can be translated into arguments by limits and Riemann integrals to yield the results more rigorously.

Note that the resulting integral is usually difficult to evaluate because the integrand involves a square root of a sum. When the curve is a straight line or a circle, it's easy. Lengths of arcs of a parabola can be computed, too. But even a curve as simple as an ellipse gives a nonelementary integral, that is, an integral that can't be evaluated in terms of the usual elementary functions that include algebraic functions, trig functions, exponential functions, and their inverses. To find arclengths for ellipses, special functions had to be created.

Example 1 (The length of a helix.)**.** A cylindrical helix is the curve you get when you wind a string around a cylinder so that each winding is a little higher on the cylinder. It's equation is

$$\mathbf{x}(t) = (a\cos t, a\sin t, bt)$$

where a is the radius of the cylinder and $2\pi b$ is how much higher on the cylinder the next winding is. Its velocity is

$$\mathbf{x}'(t) = (-a\sin t, a\cos t, b),$$

so its speed is

$$\|\mathbf{x}'(t)\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2 + b^2}.$$

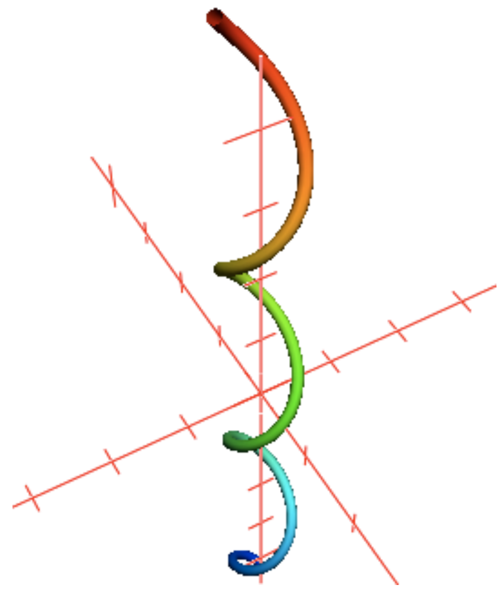


Figure 1: Helix

Since the speed is constantly $\sqrt{a^2+b^2}$, then, of course, the length L of this helix over the time interval $[0, t]$ is $L = t\sqrt{a^2+b^2}$, but we could evaluate that by an integral, too:

$$\begin{aligned} L &= \int_0^t \|\mathbf{x}'(\tau)\|\, d\tau \\ &= \int_0^t \sqrt{a^2+b^2}\, d\tau \\ &= \sqrt{a^2+b^2}\,\tau\Big|_{\tau=0}^t \\ &= t\sqrt{a^2+b^2} \end{aligned}$$

(Since the variable t is used as a limit of integration, some other variable is needed in the integral. Any variable, like τ will do.)

The arclength parameter. So if dt is the differential for t, and dx is the differential for x, then what is ds the differential for? It will be for s, but we have to figure out what s means.

Let $\mathbf{x}$ be a C^1 path in $\mathbf{R}^n$ over the time interval $[a, b]$, and assume that $\mathbf{x}'(t)$ is never $\mathbf{0}$. Let $s(t)$ denote the length of the path over the interval $[a, t]$:

$$s(t) = \int_a^t \|\mathbf{x}'\| = \int_a^t \|\mathbf{x}'(\tau)\|\, d\tau.$$

By the fundamental theorem of calculus, if we now differentiate the last integral, we get

$$\frac{ds}{dt} = s' = \|\mathbf{x}'(t)\|,$$

which says that the derivative of the arclength s is the speed. In other words, ds is the differential of the arclength s.

That makes sense when you reconsider why we defined ds in the first place, that is, when it appeared in Leibniz' differential triangle. There, ds was an infinitesimal piece of the curve.

Parameterizing a curve by its arclength. Sometimes we have a curve in $\mathbf{R}^n$ and we're looking for a parameterization of it. There are, of course, many parameterizations of a curve since a path $\mathbf{x}$ may travel down the curve at any rate and still traverse the curve. But we would like to have a standard parameterization for the curve, and a natural candidate for that is the path $\mathbf{x}$ that traverses the curve at unit speed. That would be a path $\mathbf{x}$ whose speed is 1, that is, $\|\mathbf{x}'(t)\| = 1$. You can create such a path from any path $\mathbf{x}$ (so long as $\mathbf{x}$ is C^1 and it's derivative is never $\mathbf{0}$) by reparamatrizing it, that is, by making s the independent variable instead of t. Note that

$$\mathbf{x}'(s) = \frac{d\mathbf{x}}{ds} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \mathbf{x}'(t)\ \frac{1}{ds/dt} = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}$$

In practice, reparametrizing $\mathbf{x}$ in terms of s is difficult, but not always.

Example 2 (The helix again)**.** Continue the helix example started above. For that, the arclength was easy to find, and we found length of the helix over the time interval $[0, t]$ was

$$s = s(t) = t\sqrt{a^2 + b^2}.$$

We can easily solve for t in terms of s to get

$$t = \frac{s}{\sqrt{a^2 + b^2}}.$$

Now that we know t in terms of s, we can easily reparametrize the curve $\mathbf{x}$ in terms of s, just by substituting. Since

$$\mathbf{x}(t) = (a\cos t, a\sin t, bt),$$

therefore, $\mathbf{x}(s)$, by which we mean $\mathbf{x}(t(s))$, is

$$\mathbf{x}(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}}, a\sin\frac{s}{\sqrt{a^2+b^2}}, \frac{bs}{\sqrt{a^2+b^2}}\right).$$

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