CLARK UNIVERSITY 1887

Gradient, divergence, and curl
Math 131 Multivariate Calculus
D Joyce, Spring 2014

The del operator ∇. First, we'll start by abstracting the gradient ∇ to an operator. By the way, the gradient of f isn't always denoted ∇f; sometimes it's denoted grad f.

As you know the gradient of a scalar field $f : \mathbf{R}^n \to \mathbf{R}$ is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

We can abstract this by leaving out the f to get an operator

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)$$

which, when applied to f yields ∇f. This ∇ is called the *del operator*.

We can treat this del operator like a vector itself. We can combine it with other vector operations like dot product and cross product, and that leads to the concepts of divergence and curl, respectively.

Definition 1. We define the *divergence* of a vector field $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^n$ as

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}.$$

We'll look at a couple of examples in class. As we do so, we'll develop the idea that div $\mathbf{F}(\mathbf{x})$ somehow measures the rate of flow out of the point $\mathbf{x}$, at least when $\mathbf{F}$ measures the velocity of a fluid. When a vector field $\mathbf{F}$ has 0 divergence, i.e., div $\mathbf{F}$ is constantly 0, we say $\mathbf{F}$ is *incompressible* or *solenoidal*.

Definition 2. We define the *curl* of a vector field in space, $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$, as

$$\begin{aligned}\text{curl } \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_1, F_2, F_3) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\end{aligned}$$

We'll look at a couple of examples of curl in class, too. It's harder to get a good intuition for curl, but it does say something about how much and which way a vector field swirls, or rotates. A vector field whose curl is constantly $\mathbf{0}$ is called *irrotational*.

You can take curls of plane vector fields $\mathbf{F} : \mathbf{R}^2 \to \mathbf{R}^2$, too. Just assume that the first two coordinate functions F_1 and F_2 don't depend on z and the third coordinate function F_3 is 0. Then the first two coordinates of curl $\mathbf{F}$ are 0 leaving only the third coordinate

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

as the curl of a plane vector field.

A couple of theorems about curl, gradient, and divergence. The gradient, curl, and divergence have certain special composition properties, specifically, the curl of a gradient is $\mathbf{0}$, and the divergence of a curl is $\mathbf{0}$.

The first says that the curl of a gradient field is $\mathbf{0}$. If $f : \mathbf{R}^3 \to \mathbf{R}$ is a scalar field, then its gradient, ∇f, is a vector field, in fact, what we called a gradient field, so it has a curl. The first theorem says this curl is $\mathbf{0}$. In other words, gradient fields are irrotational.

Theorem 3. If a scalar field $f : \mathbf{R}^3 \to \mathbf{R}$ has continuous second partial derivatives, then

$$\text{curl (grad } f) = \nabla \times (\nabla f) = \mathbf{0}$$

Proof. Since

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

and

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right),$$

therefore $\nabla \times (\nabla f)$ equals

$$\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}, \frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}, \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \right).$$

Since f has continuous second partials, the order that the partials are taken doesn't matter, so the last expression simplifies to $(0, 0, 0)$. Q.E.D.

Theorem 4. If a vector field $\mathbf{F} : \mathbf{R}^3 \to \mathbf{R}^3$ has continuous second partial derivatives of its coordinate functions, then

$$\text{div}\ (\text{curl}\ \mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) = \mathbf{0}.$$

Proof. First, note that $\nabla \cdot (\nabla \times \mathbf{F})$ equals

$$\tfrac{\partial}{\partial x}(\tfrac{\partial F_3}{\partial y} - \tfrac{\partial F_2}{\partial z}) + \tfrac{\partial}{\partial y}(\tfrac{\partial F_1}{\partial z} - \tfrac{\partial F_3}{\partial x}) + \tfrac{\partial}{\partial z}(\tfrac{\partial F_2}{\partial x} - \tfrac{\partial F_1}{\partial y}).$$

Since order the partials are taken doesn't matter, the expression simplifies to 0. Q.E.D.

Some of the other properties of div and curl are mentioned in the exercises for the section. First of all, they're both linear. If k is a scalar, and $\mathbf{F}$ and $\mathbf{G}$ are vector fields, then

$$\begin{aligned} \text{div}\ (k\mathbf{F}) &= k\,\text{div}\ \mathbf{F} \\ \text{div}\ (\mathbf{F} \pm \mathbf{G}) &= \text{div}\ \mathbf{F} \pm \text{div}\ \mathbf{G} \\ \text{curl}\ (k\mathbf{F}) &= k\,\text{curl}\ \mathbf{F} \\ \text{curl}\ (\mathbf{F} \pm \mathbf{G}) &= \text{curl}\ \mathbf{F} \pm \text{curl}\ \mathbf{G} \end{aligned}$$

Some version of the product rule also works for them. Here f is a scalar field, and $\mathbf{F}$ and $\mathbf{G}$ are vector fields.

$$\begin{aligned} \text{div}\ (f\mathbf{G}) &= f\,\text{div}\ \mathbf{G} + (\text{grad}\ f) \cdot \mathbf{G} \\ \text{curl}\ (f\mathbf{G}) &= f\,\text{curl}\ \mathbf{G} + (\text{grad}\ f) \times \mathbf{G} \\ \text{div}\ (\mathbf{F} \times \mathbf{G}) &= \mathbf{G} \cdot \text{curl}\ \mathbf{F} - \mathbf{F} \cdot \text{curl}\ \mathbf{G} \end{aligned}$$

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