Length, dot products, and cross products in $\mathbf{R}^3$

Math 131 Multivariate Calculus

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Length, dot products, and cross products together allow us to do geometry in three dimensions. This page is a quick review of them, not a complete study.

Length and distance. The length $\|(a_1, a_2, a_3)\|$ of a vector $\mathbf{a} = (a_1, a_2, a_3)$ (also called its norm) is defined as

$$\|(a_1, a_2, a_3)\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

The length of a vector $\mathbf{a}$ is just the distance from $\mathbf{a}$ to the origin $\mathbf{0} = (0, 0, 0)$.

The distance between two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is the length of the displacement vector $\mathbf{b} - \mathbf{a}$ between them

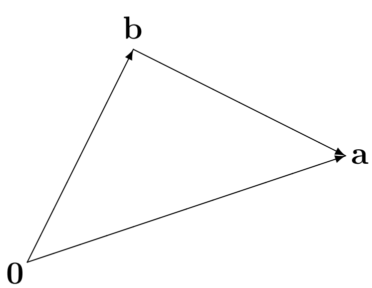
$$\|(b_1, b_2, b_3) - (a_1, a_2, a_3)\| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}.$$

Scaling a vector, that is, multiplying it by a constant c changes its length by a factor of the absolute value of c

$$\|c\mathbf{a}\| = |c|\,\|\mathbf{a}\|.$$

The triangle inequality holds for vectors

$$\|\mathbf{a} - \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|.$$



It expresses the fact that one side of a triangle $\mathbf{a} - \mathbf{b}$ is no longer than the sum of the other two $\|\mathbf{a}\| + \|\mathbf{b}\|$. See Euclid's *Elements* Proposition I.20 for Euclid's proof of this inequality.

Unit vectors. A *unit vector* is a vector whose length is 1. We can interpret unit vectors as being directions, and we can use them in place of angles since they carry the same information as an angle. Unit vectors can be identified with points on the unit sphere

$$S^2 = \{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1\}.$$

Now that we can talk about the length of a vector, we can construct a unit vector in the same direction as a given vector simply by dividing by its length. If $\mathbf{a}$ is a vector in $\mathbf{R}^3$, the unit vector in the same direction is

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|}.$$

We can rewrite that equation as

$$\mathbf{a} = \|\mathbf{a}\|\,\mathbf{u}$$

which says that a vector $\mathbf{a}$ is the product of its length and its direction.

Dot products. The dot products $\mathbf{a} \cdot \mathbf{b}$ of vectors (sometimes called inner products and denoted $\langle \mathbf{a} | \mathbf{b} \rangle$) is the sum of the products of corresponding coordinates, that is,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_1, a_2, a_3) \cdot (b_1, b_2, b_3) \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned}$$

Notice right away that we can interpret the square of the length of the vector as an inner product. Since

$$\|\mathbf{a}\|^2 = a_1^2 + a_2^2 + a_3^2,$$

therefore

$$\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}.$$

Because of this connection between norm and inner product, we can often reduce computations involving length to simpler computations involving inner products.

Inner products are commutative:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}.$$

Also, inner products distribute over addition,

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{a} \cdot \mathbf{c}),$$

and over subtraction,

$$\mathbf{a} \cdot (\mathbf{b} - \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{c}),$$

and the inner product of any vector and the $\mathbf{0}$ vector is 0

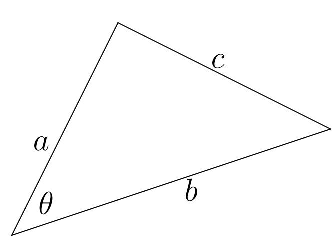
$$\mathbf{a} \cdot \mathbf{0} = 0.$$

Furthermore, dot products and scalar products have a kind of associativity, namely, if c is a scalar, then

$$(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v}).$$

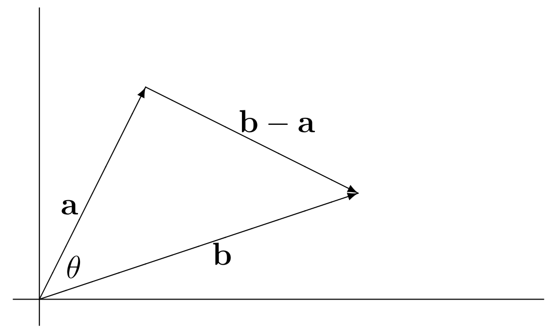
These last few statements can be summarized by saying that dot products are linear in each coordinate, or that dot products are bilinear operations.

The dot product of two vectors and the cosine of the angle between them. The law of cosines for oblique triangles says that given a triangle with sides a, b, and c, and angle θ between sides a and b,



$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Now, start with two vectors $\mathbf{a}$ and $\mathbf{b}$, and place them in the plane with their tails at the same point. Let θ be the angle between these two vectors. The vector that joins the head of $\mathbf{a}$ to the head of $\mathbf{b}$ is $\mathbf{b} - \mathbf{a}$. Now we can use the law of cosines to see that



$$\|\mathbf{b} - \mathbf{a}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta.$$

We can convert the distances to dot products to simplify this equation.

$$\begin{aligned} \|\mathbf{b} - \mathbf{a}\|^2 &= (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{b} \cdot \mathbf{b} - 2\mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{a} \\ &= \|\mathbf{b}\|^2 - 2\mathbf{b} \cdot \mathbf{a} + \|\mathbf{a}\|^2 \end{aligned}$$

Now, if we subtract $\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2$ from both sides of our equation, and then divide by -2, we get

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta.$$

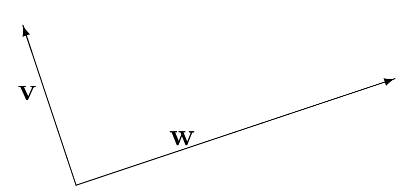
That gives us a way of geometrically interpreting the dot product. We can also solve the last equation for $\cos\theta$,

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|},$$

which will allow us to do trigonometry by means of linear algebra. Note that

$$\theta = \arccos\left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}\right).$$

Orthogonal vectors. The word "orthogonal" is synonymous with the word "perpendicular," but for some reason is preferred in many branches of mathematics. We'll write $\mathbf{w} \perp \mathbf{v}$ if the vectors $\mathbf{w}$ and $\mathbf{v}$ are orthogonal, or perpendicular.



Two vectors are orthogonal if the angle between them is 90°. Since the cosine of 90° is 0, that means

$$\mathbf{w} \perp \mathbf{v} \text{ if and only if } \langle \mathbf{w} | \mathbf{v} \rangle = 0$$

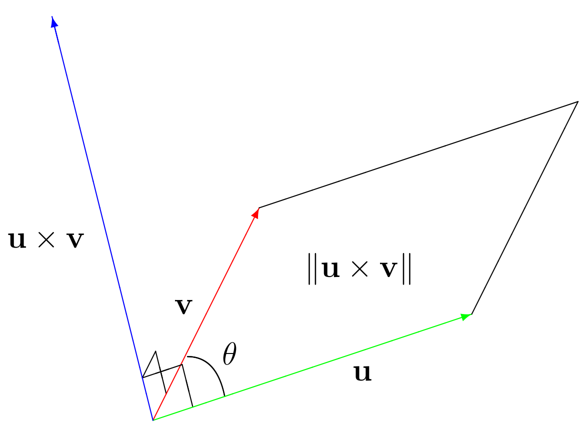
Two vectors are orthogonal, $\mathbf{a} \perp \mathbf{b}$, if and only if their dot product is 0.

Projections of one vector onto another. The projection of one vector $\mathbf{b}$ onto another vector $\mathbf{a}$ is defined to be the vector

$$\operatorname{proj}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{a}\|^2} \mathbf{a}.$$

It's the component of $\mathbf{b}$ in the direction of $\mathbf{a}$. This projection of one vector on another is useful in answering some questions about geometry using linear algebra.

The definition of cross products. The cross product $\times : \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ is an operation that takes two vectors $\mathbf{u}$ and $\mathbf{v}$ in space and determines another vector $\mathbf{u} \times \mathbf{v}$ in space. (Cross products are sometimes called outer products, sometimes called vector products.) Although we'll define $\mathbf{u} \times \mathbf{v}$ algebraically, its geometric meaning is more understandable.



The cross product $\mathbf{u} \times \mathbf{v}$ is determined by its length and its direction. It's length is equal to the area of the parallelgram whose sides are $\mathbf{u}$ and $\mathbf{v}$, and that area is the length of $\mathbf{u}$ times the length of $\mathbf{v}$ times the sine of the angle θ between them. Thus

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta$$

The direction of $\mathbf{u} \times \mathbf{v}$ will be orthogonal to the plane of $\mathbf{u}$ and $\mathbf{v}$ in a direction determined by a right-hand rule (when the coordinate system is right-handed).

The easiest way to define cross products is to use the standard unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ for $\mathbf{R}^3$. If

$$\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k},$$

and

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k},$$

then $\mathbf{u} \times \mathbf{v}$ is defined as

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

which is much easier to remember when you write it as a determinant

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{aligned}$$

Properties of cross products. There are a whole lot of properties that follow from this definition. First of all, it's anticommutative

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}),$$

so any vector cross itself is $\mathbf{0}$

$$\mathbf{v} \times \mathbf{v} = \mathbf{0}.$$

It's bilinear, that is, linear in each argument, so it distributes over addition and subtraction, $\mathbf{0}$ acts as zero should, and you can pass scalars in and out of arguments

$$\mathbf{u} \times (\mathbf{v} \pm \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \pm (\mathbf{u} \times \mathbf{w})$$

$$(\mathbf{u} \pm \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) \pm (\mathbf{v} \times \mathbf{w})$$

$$\mathbf{0} \times \mathbf{v} = \mathbf{0} = \mathbf{v} \times \mathbf{0}$$

$$c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$$

A couple more properties you can check from the definition, or from the properties already found are that $\langle \mathbf{u} \times \mathbf{v} | \mathbf{u} \rangle = 0$ and $\langle \mathbf{u} \times \mathbf{v} | \mathbf{v} \rangle = 0$. Those imply that the vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both vectors $\mathbf{u}$ and $\mathbf{v}$, and so it is orthogonal to the plane of $\mathbf{u}$ and $\mathbf{v}$.

Standard unit vectors and cross products. Interesting things happen when we look specifically at the cross products of standard unit vectors. Of course

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0},$$

since any vector cross itself is $\mathbf{0}$. But

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j},$$

and

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j},$$

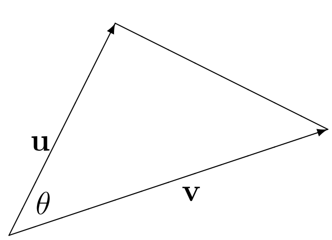
all of which follows directly from the definition.

Length of the cross product, areas of triangles and parallelograms. A direct computation (which we'll omit) shows that

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \, \sin\theta$$

where θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$.

Consider a triangle in 3-space where two of the sides are $\mathbf{u}$ and $\mathbf{v}$.

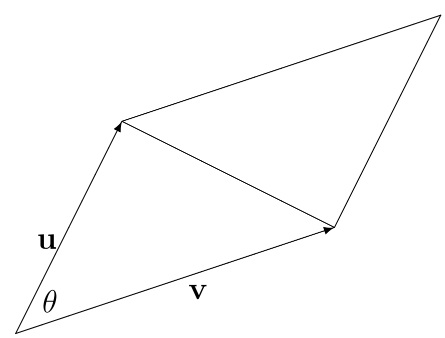


Taking $\mathbf{u}$ to be the base of the triangle, then the height of the triangle is $\|\mathbf{v}\| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Therefore, the area of this triangle is

$$\text{Area} = \tfrac{1}{2}\|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta = \tfrac{1}{2}\|\mathbf{u} \times \mathbf{v}\|.$$

(In general, the area of a any triangle is half the product of two adjacent sides and the sine of the angle between them.)

Area of a parallelogram in $\mathbf{R}^3$. Now consider a parallelogram in 3-space where two of the sides are $\mathbf{u}$ and $\mathbf{v}$.



Of course, if the triangle is doubled to a parallelogram, then the area of the parallelogram is $\|\mathbf{u} \times \mathbf{v}\|$.

Thus, the norm of a cross product is the area of the parallelgram bounded by the vectors.

We now have a geometric characterization of the cross product. The cross product $\mathbf{u} \times \mathbf{v}$ is the vector orthogonal to the plane of $\mathbf{u}$ and $\mathbf{v}$ pointing away from it in a the direction determined by a right-hand rule, and its length equals the area of the parallelgram whose sides are $\mathbf{u}$ and $\mathbf{v}$.

Note that $\mathbf{u} \times \mathbf{v}$ is $\mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ lie in a line, that is, they point in the same direction or the directly opposite directions.

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