Double integrals
Math 131 Multivariate Calculus
D Joyce, Spring 2014

Volumes as integrals. Recall from calculus of one variable that we had a general principle to find the volume that's bounded between planes $x = a$ and $x = b$. The volume V of that region is given by the integral

$$V = \int_a^b A(x)\,dx$$

where $A(x)$ is the area of the cross section at x. For instance, the disk and washer methods find the volumes of solids of revolution where the cross sections are circles and annuli.

We'll start with that principle as a basis for double integrals.

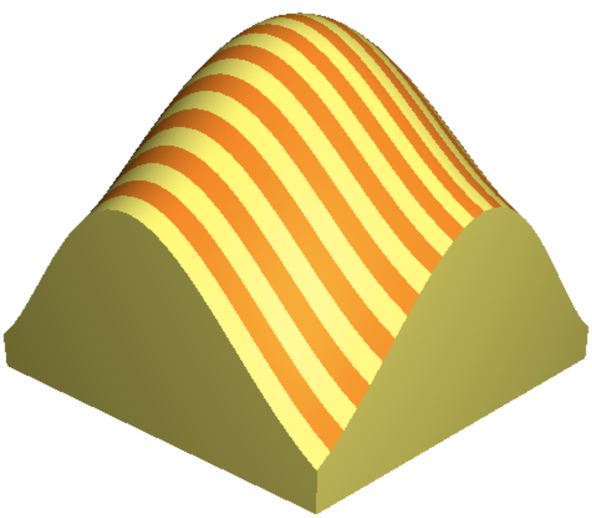

Figure 1: Integration over a rectangle

Integrating over rectangles. Now suppose each cross section of the region is not only bounded between the planes $x = a$ and $x = b$, but also bounded between the planes $y = c$ and $y = d$. That is to say, the region is entirely over (or under) the rectangle $R = [a, b] \times [c, d]$ in the (x, y)-plane. Then the area of the cross section at x is also given by an integral, namely,

$$A(x) = \int_{y=c}^{d} C(x, y)\,dy$$

where $C(x, y)$ is the length of the is the length of the cross sectional line at (x, y), that is, the length of the line above the point (x, y) inside the volume. Thus, the volume of the region is an integral of an integral:

$$V = \int_{x=a}^{b} \left(\int_{y=c}^{d} C(x, y)\,dy \right) dx.$$

Usually the parentheses are left out and the variables aren't mentioned in the limits of integration

$$V = \int_a^b \int_c^d C(x, y)\,dy\,dx.$$

When that's done, the last differential dx indicates that the first integral has limits in terms of the associated variable, that is, $a \le x \le b$, while the next to the last differential dy indicates that the next integral has limits in terms of the next associated variable, $c \le y \le d$.

Typically, we're interested in the area under the surface $z = f(x, y)$ and above the (x, y)-plane, so the cross sectional length $C(x, y)$ equals $f(x, y)$.

We can also find the volume V by first intersecting the region by planes with a fixed y value rather than a fixed x value, then second by planes with a fixed x value rather than a fixed y value. You get the same volume, but with a different double integral

$$V = \int_{y=c}^{d} \int_{x=a}^{b} C(x, y)\,dx\,dy.$$

This exchange of the order of integration is called Fubini's theorem. It works so long as (1) you integrate over a rectangle $R = [a, b] \times [c, d]$, and (2) the length of the cross section $C(x, y)$ is a continuous function of x and y. There are lots of other weaker conditions than continuity that can replace (2), but most of the functions we consider are continuous, so that condition will do for us.

Formal definition and properties of double integrals. Integrals over a rectangle R

$$\iint_R f(x, y)\, dx\, dy$$

can be defined formally in terms of Riemann sums as was done for single integrals and the analogous results follow. In this definition f doesn't have to have positive values but can also have negative values. Of course, if f has only negative values, then the integral is negative, too.

The usual properties that hold for single integrals, such as linearity, monotonicity, etc., also hold for double integrals.

Integrals over regions other than rectangles. Some times we need integrals of functions defined over regions other than rectangles. By extending these functions be defining them to be 0 outside their original domain, the domains can be extended to rectangles, and then the previous double integrals work.

But, in practice, there are easier ways. The text describes a couple of these nonrectangular domains D. For the first type, the domain $D \subseteq \mathbf{R}^2$ is bounded on the left by the line $x = a$, on the right by the line $x = b$, below by the function $y = \gamma(x)$, and above by the function $y = \delta(x)$. In that case, you can find the double integral as an iterated integral

$$\iint_D f = \int_{x=a}^{b} \int_{y=\gamma(x)}^{\delta(x)} f(x, y)\, dy\, dx.$$

We'll look at an example in class.

Another type of region is bounded above and below by straight lines but to the left and right by curves, and for that type of region, a similar double integral works where the order of x and y are reversed.

For regions that aren't of either of these types, you can break the region into subregions of those two types and add their integrals together.

Incidentally, you can find the area of a region $D \subseteq \mathbf{R}^2$ by a double integral of the constant function 1:

$$\text{Area}(D) = \iint_D 1\, dx\, dy.$$

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