CLARK UNIVERSITY 1887

# Functions of several variables

Math 131 Multivariate Calculus

D Joyce, Spring 2014

The subject matter of this course concerns functions $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$, but first we'll look at some standard concepts that apply to any function $f : X \to Y$ from one set $X$ to another set $Y$.

**Domains and codomains.** You're familiar with the concept of function from calculus and from linear algebra.

In calculus of one variable, the domain $X$ of a function $f : X \to Y$ was a subset of the real numbers $\mathbf{R}$, typically an interval, and the codomain $Y$ was $\mathbf{R}$.

In linear algebra, the domain of a linear transformation $T : V \to W$ was one vector space $V$, and the codomain $W$ was another vector space. We also used the term *linear operator* when $V$ and $W$ were the same vector space.

In this course we'll look at functions $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ whose domain is a subset of $n$-space, $\mathbf{R}^n$ and whose codomain is $m$-space, $\mathbf{R}^m$. Three special cases of particular interest are

- Scalar fields in $n$-space. These are functions $f : \mathbf{R}^n \to \mathbf{R}$ when $m = 1$.
- Paths in $m$-space. These are functions $\mathbf{f} : \mathbf{R} \to \mathbf{R}^m$ when $n = 1$.
- Vector fields in $n$-space. These are functions $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ when $m = n$.

When the codomain $\mathbf{R}^m$ is just the real numbers $\mathbf{R}$, that is, $\mathbf{R}^1$, we say $f$ is *scalar-valued*. But when $m > 1$, we say $f$ is *vector-valued*, and in that case we use boldface for the name of the function, $\mathbf{f}$, just as we use boldface for vectors.

**Onto functions (surjective), one-to-one functions (injective), one-to-one correspondences (bijective), range.** These terms mean the same thing in multivariate calculus as they in calculus of a single variable. Be sure you're familiar with the following concepts for a function $f : X \to Y$.

The function $f$ is said to be *onto* or *surjective* if every element of $Y$ is the image $f(x)$ of at least one element $x \in X$, that is,

$$\forall y \in Y, \exists x \in X, f(x) = y.$$

The function $f$ is said to be *one-to-one* or *injective* if no two distinct elements of $X$ have the same image under $f$, that is, $f(x_1) = f(x_2)$ only when $x_1 = x_2$. That can be stated as

$$\forall y \in Y, \exists \text{ at most one } x \in X, f(x) = y.$$

Those functions that are both one-to-one and onto are called *one-to-one correspondences* or *bijective* functions.

$$\forall y \in Y, \exists \text{ exactly one } x \in X, f(x) = y.$$

The quantifier indicating that there is exactly one $x$ satisfying a condition is usually written $\exists! x$.

The function $f : X \to Y$ is bijective if and only if there is an *inverse function* $f^{-1} : Y \to X$, a function such that

$$\forall x \in X, \forall y \in Y, y = f(x) \text{ iff } x = f^{-1}(y).$$

Note that there can be at most one inverse function. An equivalent condition for $f^{-1}$ to be inverse to $f$ is that the two compositions are both identity functions, that is, $f^{-1} \circ f$ is the identity function on $X$, and $f \circ f^{-1}$ is the identity function on $Y$.

The *image* of a function $f : X \to Y$ is the subset of $Y$ consisting of all the elements of the form $f(x)$ for some $x \in X$. The image of a function $f$ is often denoted $f(X)$. Note that $f$ is onto exactly when the range is all of $Y$.

Note: the term *range* is used ambiguously in mathematics. Some people use range to mean the codomain $Y$ of a function $f : X \to Y$, but others use it to mean the image $f(X)$. We'll use the terms codomain and image since those terms are unambiguous.

**Overloading symbols for variables and functions.** We'll be working with several functions at the same time, each with many coordinates. Although it would be nice to have a different symbol for each function and variable, it becomes hard to keep track of so many symbols.

In calculus of one variable, when $x$ was a function of $t$, we used another symbol, $f$ to describe that function, and we used the functional notation $x = f(t)$. An alternative would be to use the same symbol $x$ for both the variable $x$ and the function $f$. We would then write $x = x(t)$. You could read that equation as saying that $x$ is a variable that, parenthetically, depends on $t$.

Overloading the same symbol $x$ with two meanings like that can cause confusion, but it also aids in understanding since we won't have to remember that the symbol $x$ is related to the symbol $f$ somehow.

(I've often said that there there are two principles that lead to mathematical insight: confusion and forgetfulness. I get more insightful every day. lol)

**Curves and paths.** We discussed curves and paths before. A path is a function $\mathbf{x} : \mathbf{R}^n \to \mathbf{R}^m$ where $n = 1$. Typically the single independent variable is taken to be $t$. It's the variable that parameterizes the path. $\mathbf{x}(t)$ is the location in $\mathbf{R}^m$ of the point at time $t$.

We'll distinguish between a path and the curve it travels. The curve is the image of this path, that is, a subset of $\mathbf{R}^m$.

**Scalar fields.** A scalar-valued function $f : \mathbf{R}^n \to \mathbf{R}$ is also called a *scalar field* on $\mathbf{R}^n$.

Lets start by considering the case when $n = 2$. We'll be able to visualize that case. There are three different ways to do that, either with a graph of the function, as a field of scalars, and as level curves.

The graph of a function $f : \mathbf{R}^2 \to \mathbf{R}$ is just a surface in 3-space whose equation is $z = f(x, y)$. It's harder to draw these surfaces, but there are plenty of computer programs that help us with such surfaces.

**Example 1.** Figure 1 shows a computer-generated image for the graph $z = x \ln(y^2 + 1)$. Note that $z = 0$ when either $x = 0$ or $y = 0$, so the $y$-axis and the $x$-axis are both within this surface. In fact, the union of those two axes form the contour at level 0. The *contour curve* of $f$ at height $c$ is the intersection of the plane $z = c$ at height $c$ above the $xy$-plane with the graph $z = f(x, y)$ of the function.

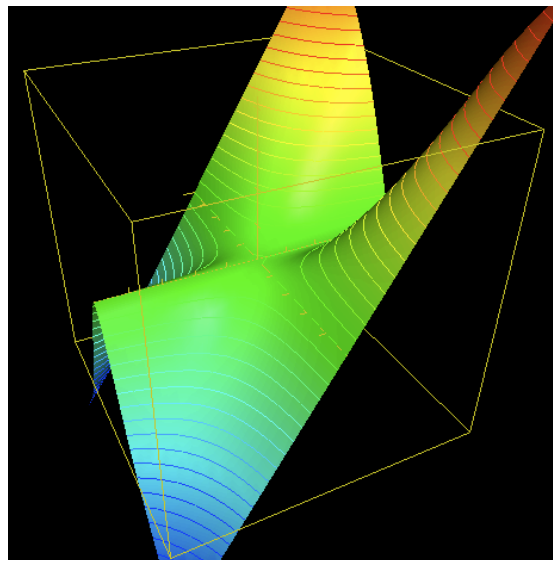

Figure 1: The graph $z = x \ln(y^2 + 1)$

The surface has some other contour curves drawn on it. Notice that $\ln(y^2+1)$ is always greater than or equal to 0, so the part of the graph $z = x \ln(y^2 + 1)$ where $x$ is positive lies above the $xy$-plane, and the part where $x$ is negative lies below it.

Another way to visualize a function $f : \mathbf{R}^2 \to \mathbf{R}$ is to use *level curves* as shown in figure 2. A level curve is just a contour curve projected down to the $xy$-plane, in other words, the graph of the equation $c = f(x, y)$ in the $xy$-plane. (Warning: what is called here a level curve is commonly called a contour curve elsewhere.)
Here you see some level curves for the same function $f(x, y) = x \ln(y^2 + 1)$.

The colors also indicate the value of the function as they did in the previous image. Green for near 0, yellow and red for positive values, and blue and violet for negative values. Rather than using

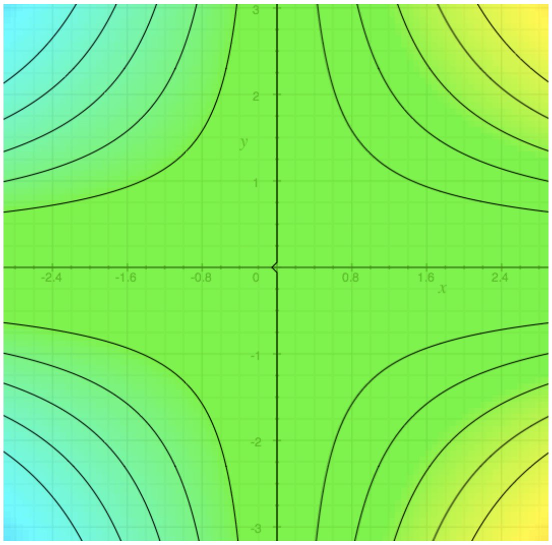

Figure 2: Level curves for $x \ln(y^2+1)$

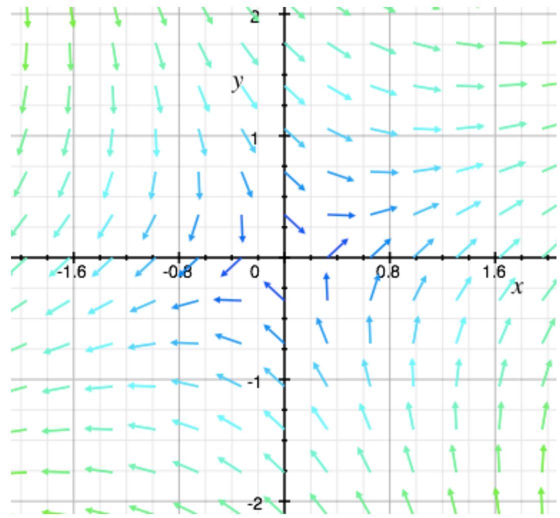

Figure 3: Vector field $(x+y, x-y)$

colors, you could actually place numbers indicating the values at selected points. In other words, scalar field on the plane associates at each point a numerical value.

You should be familiar with a few standard surfaces that frequently come up such as *quadric surfaces*. They're the surfaces described by quadratic equations

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx$$
$$+Gx + Hy + Ij + K = 0.$$

where the capital letters are constants, and at least one of the first six of them is not zero. You can see them and other surfaces at The Gallery of Surfaces.

Scalar fields in $\mathbf{R}^3$ can be imagined if you associate at each point a numerical value. They can't be drawn very well, however, and neither can their graphs.

**Vector fields.** A *vector field* on $\mathbf{R}^n$ is a vector-valued function $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$ where the domain and the codomain have the same dimension $n$. The general theory applies to the more general vector-valued functions $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ where the dimensions of the domain and codomain don't have to be the same.

A vector-valued function $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$ is usually described using its $m$ scalar-valued *component* functions $f_1, f_2, \ldots, f_m$ by

$$\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})).$$

Note that each scalar-valued component function $f_i : \mathbf{R}^n \to \mathbf{R}$ takes as an argument an $n$-vector and returns a scalar.

**Example 2.** Consider the vector field on $\mathbf{R}^2$ illustrated in figure 3. It's described by the function $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^2$ where

$$\mathbf{f}(x, y) = (x+y, x-y).$$

The first component function of $\mathbf{f}$ is $f_1(x, y) = x+y$, while the second component function is $f_2(x, y) = x-y$.

You can visualize this vector field if at each point $(x, y)$ in the plane you attach a vector whose value is $(x+y, x-y)$. For instance, the vector attached to $(0, 1)$ is $(1, -1)$ pointing down and right, while the vector attached to $(1, 0)$ is $(1, 1)$ pointing up and right.

Taken together, all the vectors in the vector field appears as a flow in the plane. We'll use that analogy to understand vector fields.

Vector fields in 3-space can be understood in the same way, but they can't be drawn.

Math 131 Home Page at
`http://math.clarku.edu/~djoyce/ma131/`