Gallery of Surfaces
Math 131 Multivariate Calculus
D Joyce, Spring 2014

There are lots of sufaces we'll use in this course. Some you're very familiar with like spheres, cylinders, cones, and tori. (Tori is the plural of torus, torus being Latin.)

Others that you may not be so familiar with are the quadric surfaces which include ellipsoids, elliptic paraboloids, hyperbolic paraboloids, and hyperboloids. Quadric surfaces are those surfaces which are solutions to quadratic equations in x, y, and z. Euler included a systematic study of the quadric surfaces in the second volume of his *Introductio in analysin infinitorum* in 1748.

Also illustrated here are the catenoids and helicoids, which are minimal surfaces, and the nonorientable surfaces known as Möbius strips.

This is a short gallery of them along with some historical tidbits and useful mathematical facts. There are many more interesting surfaces with fascinating properties.

The Sphere. The mathematical study of spheres is ancient. Euclid defined them as by revolving a semicircle about its diameter in his *Elements* Book XI Definition 14 and studied them in Books XII and XIII.

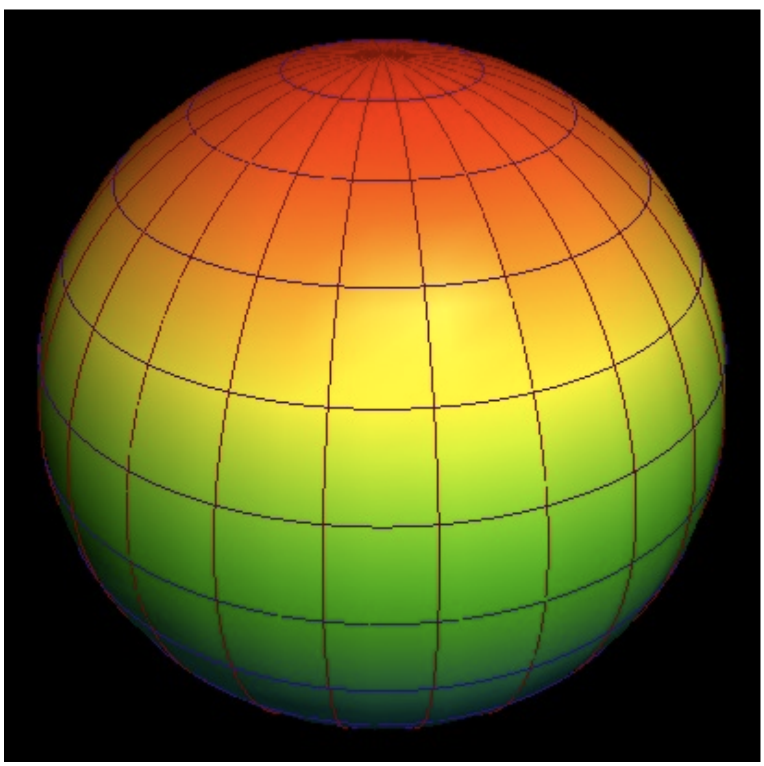

Figure 1: The sphere

The equation in rectangular coordinates for a sphere of radius r centered at the origin is

$$x^2 + y^2 + z^2 = r^2.$$

A standard parameterization of the sphere is in terms of longitude and latitude. The longitude θ can vary from 0 to 2π (or from $-\pi$ to π if you prefer), and the latitude ϕ can vary from $-\pi/2$ to $\pi/2$. (Sometimes some other variable than latitude is used, for instance, the angle from a pole in which case the angle varies from 0 to π.) For a sphere of radius r centered at the origin the parameterization is

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta\cos\phi \\ r\sin\theta\cos\phi \\ r\sin\phi \end{bmatrix}}$$

These are the same formulas for spherical coordinates for $\mathbf{R}^3$ except that for spherical coordinates, r is a variable, not a fixed radius.

Curvature. The curvature, that is the Gaussian curvature, of a surface is an intrinsic property of the surface. The curvature of most surfaces varies from point to point, but because of the symmetry of the sphere is it the same everywhere. The sphere is a surface with constant positive curvature.

The Cylinder. Like the sphere, the cylinder is ancient. See Euclid's XI. Def. 14.

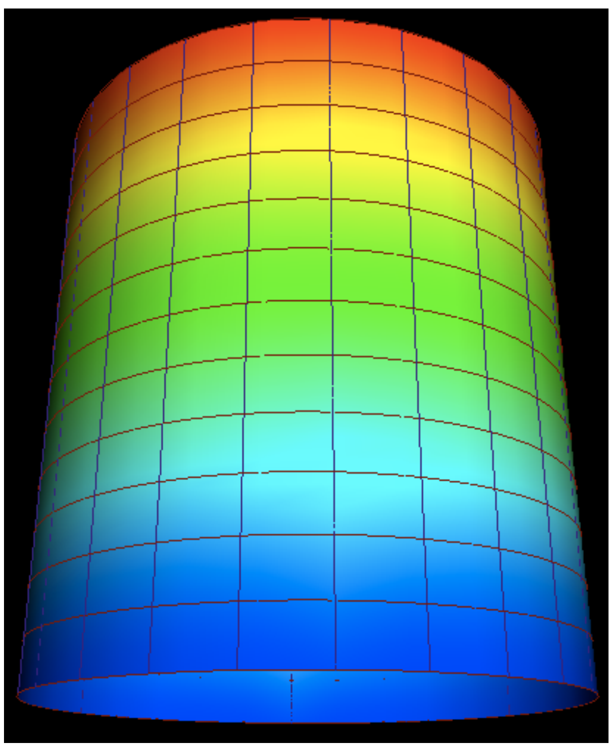

Figure 2: The cylinder

The equation in rectangular coordinates for a cylinder of radius r and whose axis is the z-axis is

$$x^2 + y^2 = r^2.$$

A standard parameterization of the cylinder is in terms of height and longitude. The longitude θ can vary from 0 to 2π, and z denotes the height. With this parameterization, the axis of the cylinder is the z-axis. The cylinder's radius is r.

The curvature of the cylinder is 0, the same as a plane, since the cylinder can be laid flat, at least when it's cut open. The cone, mentioned next, also has 0 curvature.

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ z \end{bmatrix}}$$

These are the same formulas for cylindrical coordinates for $\mathbf{R}^3$ except that for cylindrical coordinates, r is a variable, not a fixed radius.

The cylinder is an example of a single-ruled surface. Through each point there is one straight line which lies on the cylinder.

This is a right circular cylinder. The axis of the cylinder is at right angles to the plane that contains a circular cross section. There are variants of this cylinder that aren't right. If the angle is not a right angle, the resulting surface is an oblique circular cylinder. There are also variants that aren't circular if you make the cross section some other figure than a circle.

The Cone. The cone was the third solid Euclid discussed in the *Elements*. See XI. Def. 18.

A standard right circular cone has the equation $z^2 = x^2 + y^2$. It can be stretched or squeezed in various directions by dividing the variables by constants to get more general cones with the equations

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

You can parameterize these by

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} az\cos\theta \\ bz\sin\theta \\ cz \end{bmatrix}}$$

where z is the height and θ is the angle about the z-axis.

Cones, like cylinders, are singly ruled surfaces.

Conic sections. Conic sections are plane curves which are the intersections of a cone with a plane. The plane is meant not to intersect at the vertex of the cone. If the intersection meets both parts of the cone, then a hyperbola results; if it cuts across one part of the cone, then an ellipse results (for this purpose a circle is to be considered a special case of an ellipse); but if cuts only one part of the cone but not all the way across it, then a parabola results.

The Torus. A torus is another surface of revolution. It's found by rotating a circle about a line outside it.

There are two radii that determine a torus. One is the radius b of the circle being rotated, the other is the distance a from the center of that circle to the line it's being rotated about. Note that $a > b$. (When $a = b$ the resulting suface is sometimes called a *horned* torus. It has a singularity at the origin.) If the original circle is in the xz plane, with its center on

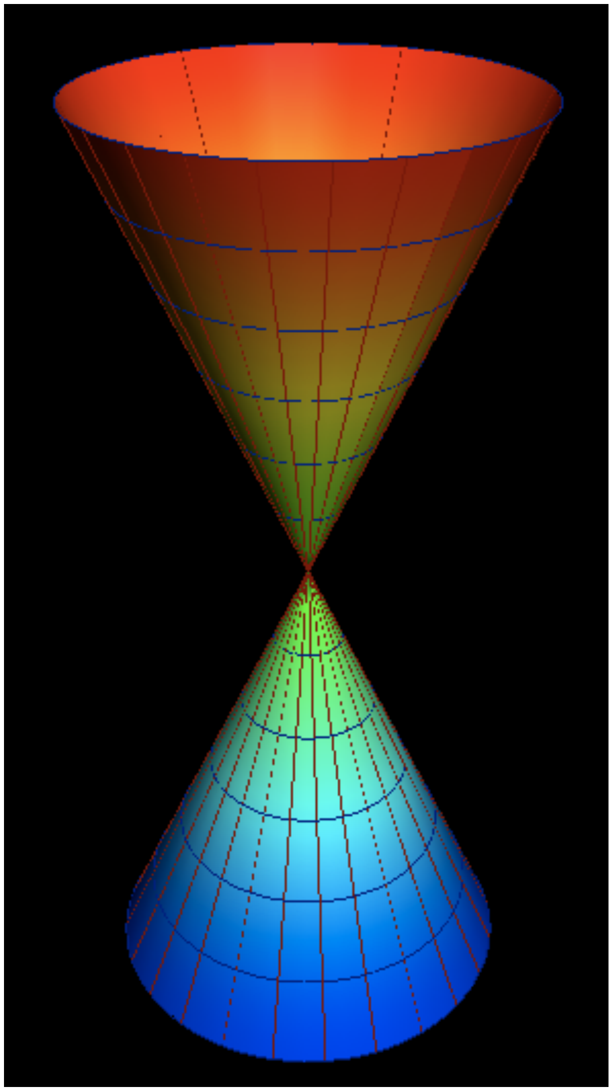

Figure 3: The cone

the x-axis, and it's being rotated about the z-axis, then an equation for the torus is the fourth-degree equation

$$(x^2 + y^2 - a^2)^2 + z^2 = b^2.$$

You can parameterize these by

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (a + b\cos\phi)\cos\theta \\ (a + b\cos\phi)\sin\theta \\ c\sin\phi \end{bmatrix}}$$

where both ϕ and θ vary from 0 to 2π.

Toric or spiric sections. Intersections of planes with tori are called toric sections or spiric sections. The older name for a torus was a spira, or speira, meaning coil or or spiral. Examples of these fourth-degree curves include the the lemniscate of Bernoulli, hippopedes, and Cassini ovals.

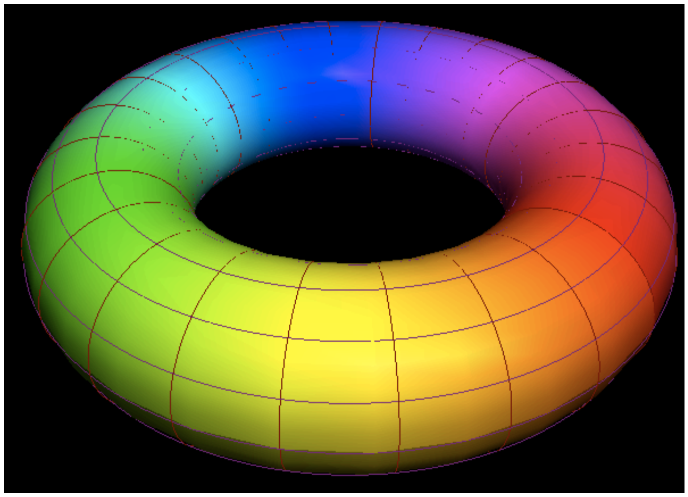

Figure 4: The torus

The Ellipsoid. Ellipsoids are simply flattened and stretched spheres with equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

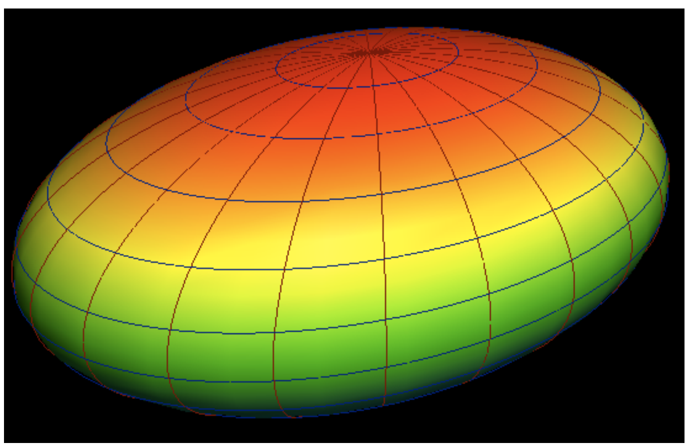

Figure 5: The Ellipsoid

If two of the constants a, b, and c are the same, they're called *spheroids*. Spheroids are surfaces of revolution that can be obtained by rotating an ellipse around its major or minor axis. If it's rotated around the major axis, the spheroid is long and thin, called a *prolate spheroid*, but if it's rotated a around its minor axis, it's short and fat, called an *oblate spheroid*.

You can parameterize ellipsoids by slightly modifying the parameterization of the sphere

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a\cos\theta\cos\phi \\ b\sin\theta\cos\phi \\ c\sin\phi \end{bmatrix}}$$

where ϕ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, and θ varies from 0 to 2π.

The Hyperboloids. Hyperboloids have the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \pm 1.$$

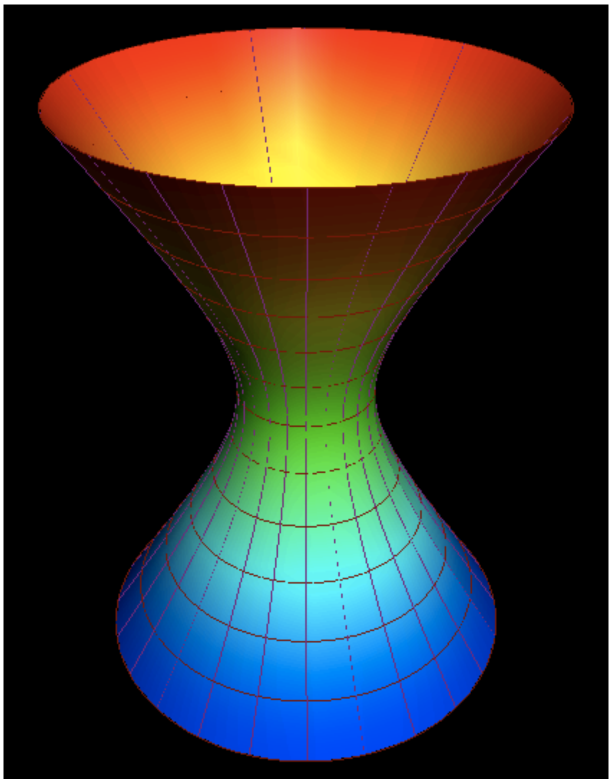

Figure 6: The Hyperboloid of one sheet

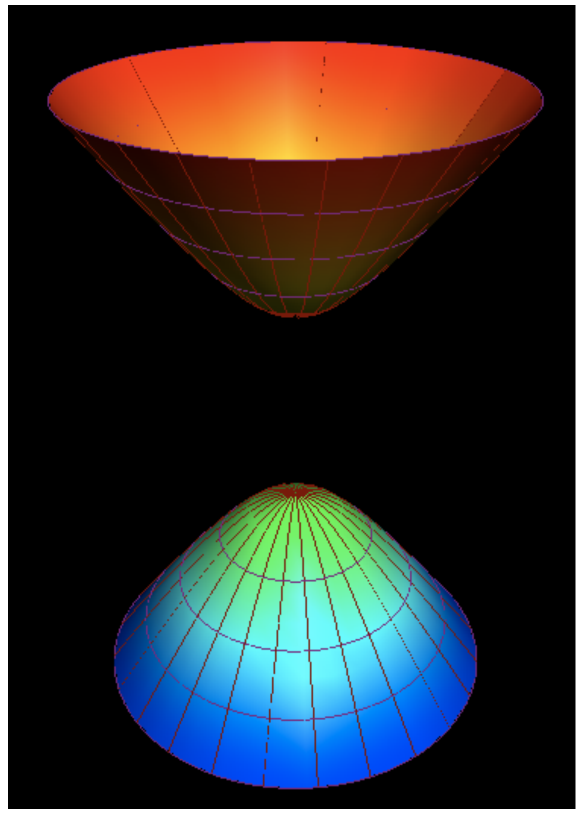

Figure 7: The Hyperboloid of two sheets

When $a = b$, the hyperbolas are circular. They're surfaces of revolution that are obtained by rotating hyperbolas about one of their axes of symmetry. When the hyperbola is rotated about the axis that doesn't meet the hyperbola, a hyperboloid of one sheets results. But when it's rotated about the other axis that meets both curves that make up the hyperbola, then a hyperboloid of two sheets results.

The hyperboloid of one sheet with the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} + 1$ can be parameterized by

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a\sqrt{t^2+1}\cos\theta \\ b\sqrt{t^2+1}\sin\theta \\ ct \end{bmatrix}}$$

where θ varies from 0 to 2π and t is a parameter.

Similarly, the hyperboloid of two sheets with the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1$ can be parameterized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a\sqrt{t^2-1}\cos\theta \\ b\sqrt{t^2-1}\sin\theta \\ ct \end{bmatrix}$$

The hyperboloid of one sheet is a doubly ruled surface. Through each its points there are two lines that lie on the surface.

Both kinds of circular hyperboloids as well as the cone can be included in one family of surfaces by modifying their defining equations slightly. Consider the equations

$$x^2 + y^2 = z^2 + e$$

where e is a constant. You get a cone when $e = 0$, a hyperbola of one sheet when $e > 0$, and a hyperbola of two sheets when $e < 0$.

The Elliptic Paraboloid. Some of the cross sections of the elliptic paraboloid are ellipses, others are paraboloids.

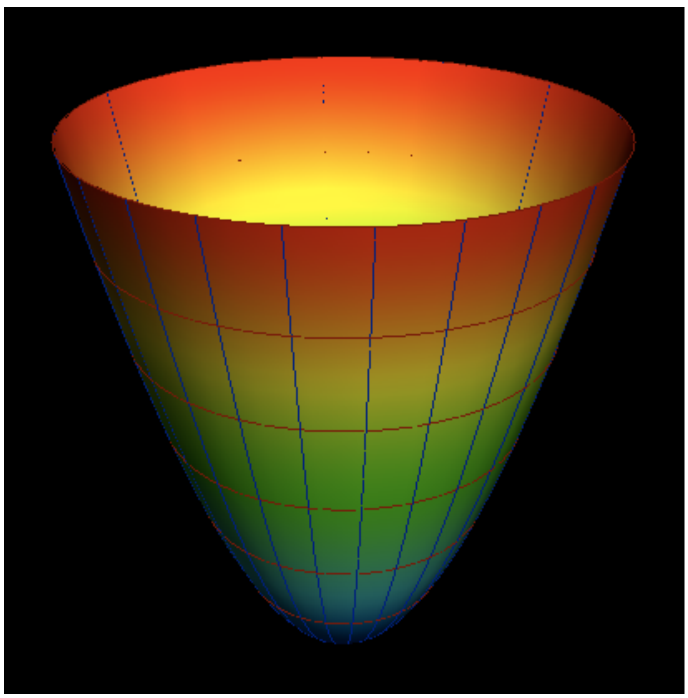

Figure 8: The Elliptic Paraboloid

The elliptic paraboloid is defined by the equation

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

When $a = b$, it's a circular paraboloid, also called a paraboloid of revolution.

It can be parameterized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} at\cos\theta \\ bt\sin\theta \\ ct^2 \end{bmatrix}$$

The circular paraboloid has an interesting focal property. If the surface is mirrored and a light source placed at its focus, then the light rays will form a parallel beam of light. Likewise a beam of light can be collected by a paraboloid to its focus.

The Hyperbolic Paraboloid. Some of the cross sections of the hyperbolic paraboloid are hyperbolas, others are paraboloids.

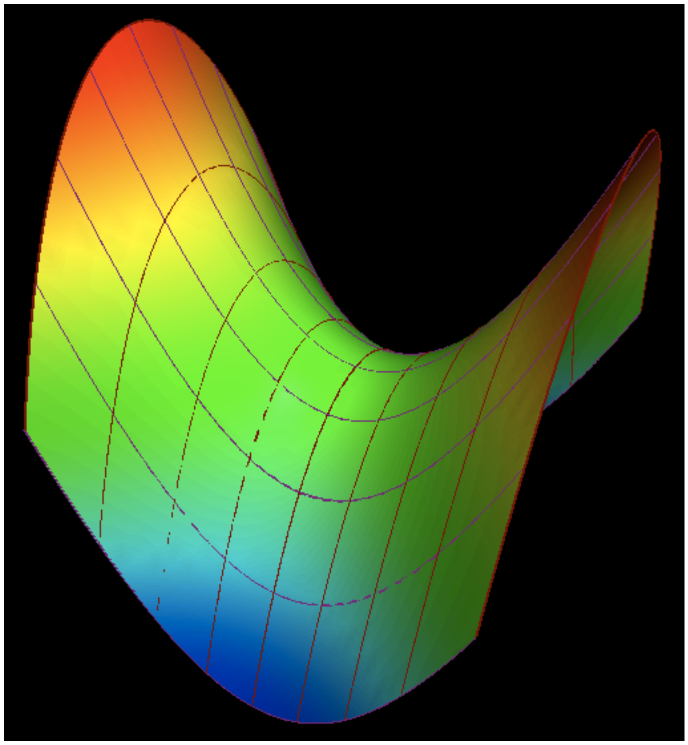

Figure 9: The Hyperbolic Paraboloid

It has the equation

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

and can be parameterized several ways including

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} as \\ bt \\ c(s^2 - t^2) \end{bmatrix}}$$

Like the hyperboloid of one sheet, the hyperbolic paraboloid is a doubly ruled surface. Through each its points there are two lines that lie on the surface.

The hyperbolic paraboloid is a surface with negative curvature, that is, a saddle surface. That's because the surface does not lie on one side of the tangent plane at a point like it would for a surface with positive curvature; instead part of the surface lies on one side of the tangent play, and part lies on the other.

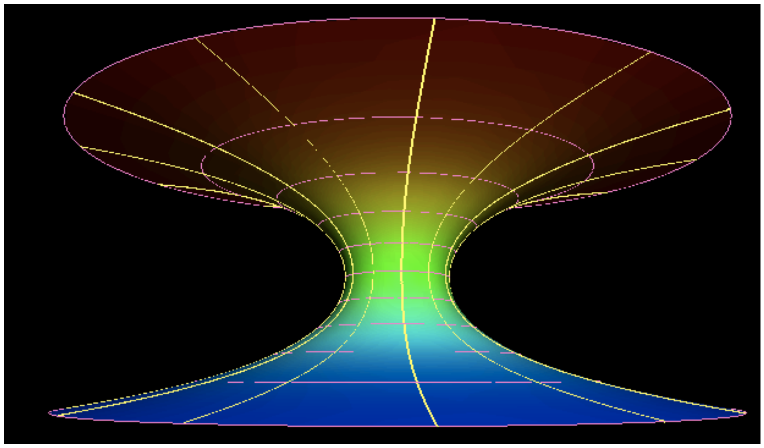

Figure 10: The Catenoid

The Catenoid. The catenoid is a surface of revolution based on a catenary curve rotated around its directrix.

A catenary is the graph of the hyperbolic cosine, also called a catenary curve. It's called a catenary because it has the shape of a chain (Latin *catena*) when held by its ends.

It can be parameterized by t and z as

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c\cosh\frac{z}{c}\,\cos\theta \\ c\cosh\frac{z}{c}\,\sin\theta \\ z \end{bmatrix}}$$

The catenoid looks a lot like the hyperboloid of one sheet since the catenary curve looks a lot like a hyperbola. The catenoid has a nice property that a hyperboloid doesn't have, however, and that's that the catenoid is a minimal surface.

Minimal surfaces. A minimal surface is a surface that minimizes area for a given boundary. Examples are soap films held by frames (but bubbles aren't allowed). Lagrange defined the term, but Meusnier found the first two nonplanar ones in 1762, namely, the catenoid and the helicoid.

The Helicoid. A helicoid is a nonalgebraic surface like a spiral staircase. It winds around an axis.

The equation for a helicoid in rectangular coordinates is the transcendental equation

$$\frac{y}{x} = \tan az$$

where a is a constant. It can be parameterized by t and z as

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t\cos az \\ t\sin az \\ z \end{bmatrix}}$$

Like the catenoid, the helicoid is a minimal surface. It is also a singly ruled surface. Through each point there is one straight line which lies on the helicoid.

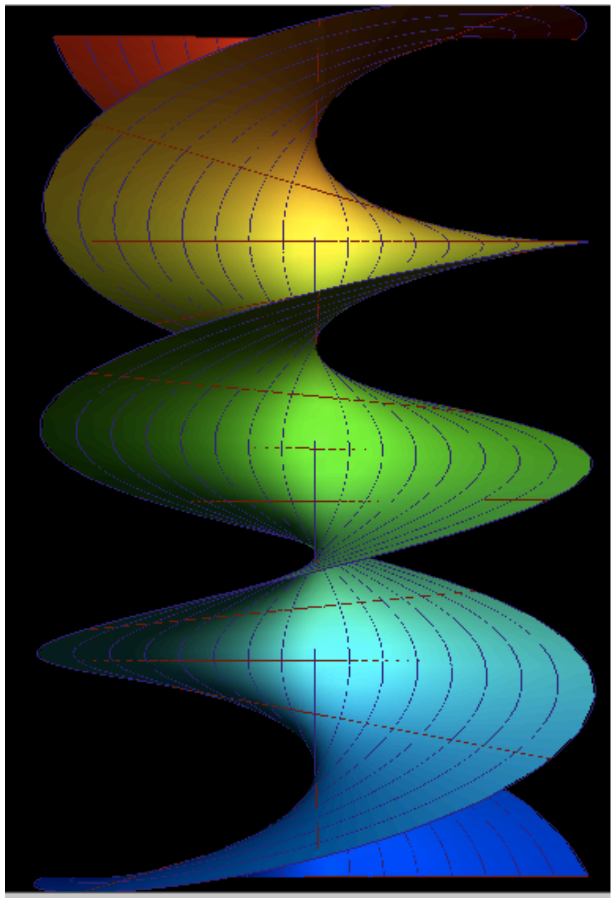

Figure 11: The Helicoid

The Möbius strip. A Möbius strip is a nonorientable surface. That is to say, no orientation can be assigned consistently to the entire surface unlike all the surfaces mentioned above. A cylinder can be constructed fron a rectangle by attaching one side to the other, but if a half twist is applied before the attachment, then a Möbius strip results.

This particular Möbius strip is parameterized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta + s\cos\frac{\theta}{2}\ \cos u \\ \sin\theta + s\cos\frac{\theta}{2}\ \sin u \\ s\sin\frac{\theta}{2} \end{bmatrix}$$

where s varies from $-\frac{1}{4}$ to $\frac{1}{4}$, and θ varies from 0 to 2π.

Math 131 Home Page at `http://math.clarku.edu/~djoyce/ma131/`

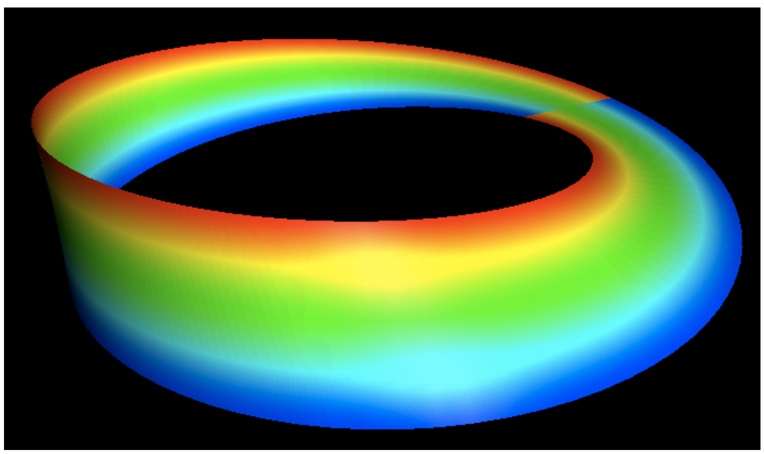

Figure 12: The Möbius Strip