Gauss's theorem
Math 131 Multivariate Calculus
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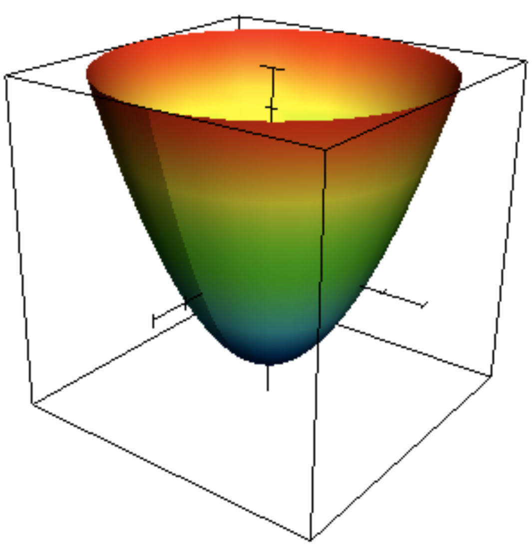
The statement of Gauss's theorem, also known as the divergence theorem. For this theorem, let D be a 3-dimensional region with boundary ∂D. This boundary ∂D will be one or more surfaces, and they all have to be oriented in the same way, away from D. Let $\mathbf{F}$ be a vector field in $\mathbf{R}^3$. Gauss' theorem equates a surface integral over ∂D with a triple integral over D. It says that the integral of $\mathbf{F}$ over ∂D equals the divergence of $\mathbf{F}$ over the region D: $\displaystyle\iint_{\partial D} \mathbf{F}\cdot d\mathbf{S} = \iiint_D \nabla\cdot\mathbf{F}\,dV.$

An interpretation of Gauss's theorem. If $\mathbf{F}(\mathbf{x})$ is the velocity of a fluid at $\mathbf{x}$, then Gauss's theorem says that the total divergence within the 3-dimensional region D is equal to the flux through the boundary ∂D. The divergence at $\mathbf{x}$ can be thought of the rate of expansion of the fluid at $\mathbf{x}$.

Example 1. Let D be the region

$$D = \{(x,y,z)\,|\,x^2+y^2+1 \le z \le 5\}.$$



The surface $x^2+y^2+1=z$ is a paraboloid opening upward (positive z being upward) with vertex on the z-axis at $z=1$. Above that surface and below the plane $z=5$ lies the 3-dimenional region D. The top surface of D is a circle of radius 2. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y,z) = (x^2, y, z).$$

We'll verify Gauss's theorem.

First, let's find $\iiint_D \nabla\cdot\mathbf{F}\,dV$, the triple integral of the divergence of $\mathbf{F}$ over D.

The divergence of $\mathbf{F}$ is

$$\nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2x+2.$$

If we interpret $\mathbf{F}(x,y,z)$ as the velocity of a flow of a fluid, than that flow has a positive divergence for $x > -1$ and negative divergence for $x < -1$. It's expanding in the first case; contracting in the second. So, in most of the paraboloid D the fluid is expanding.

We'll use cylindrical coordinates to evaluate the triple integral.

$$\begin{aligned}
&\iiint_D \nabla\cdot\mathbf{F}\,dV \\
&= \iiint_D (2x+2)\,dV \\
&= \iiint_D (2r\cos\theta+2)\,r\,dr\,d\theta\,dz \\
&= \int_0^2\int_{1+r^2}^5\int_0^{2\pi} (2r^2\cos\theta+2r)\,d\theta\,dz\,dr \\
&= \int_0^2\int_{1+r^2}^5 4\pi r\,dz\,dr \\
&= \int_0^2 4\pi r(4-r^2)\,dr = 16\pi
\end{aligned}$$

Since this integral of the divergence is positive, overall the fluid is expanding.

Now, let's go on to the harder task of evaluating the surface integral $\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S}$. The surface ∂D comes in two parts. One is the top disk S_1 at height $z = 5$ and radius 2. The other is the parabolic surface S_2. We can parametrize both of them over the domain $D' = \{(s,t) \,|\, s^2 + t^2 < 4\}$. A parametrization of S_1 is

$$\mathbf{X}_1(s,t) = (s,t,5),$$

while a parametrization of S_2 is

$$\mathbf{X}_2(s,t) = (s,t,s^2+t^2+1).$$

The normal vector for $\mathbf{X}_1$ is $\mathbf{N}_1 = (0,0,1)$ since it's a flat horizontal plane. We'll compute the normal vector for S_2.

$$\mathbf{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2s \\ 0 & 1 & 2t \end{vmatrix} = (-2s, -2t, 1).$$

Actually, there's a problem here, since the normal vector $\mathbf{N}_2$ points in toward the 3-dimensional region D. That means we'll have

$$\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{X}_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{\mathbf{X}_2} \mathbf{F} \cdot d\mathbf{S}$$

where the minus sign takes care of the direction of $\mathbf{N}_2$. We'll compute the two surface integrals.

$$\begin{aligned} & \iint_{\mathbf{X}_1} \mathbf{F} \cdot d\mathbf{S} \\ = \; & \iint_{\mathbf{X}_1} \mathbf{F} \cdot \mathbf{N}\, ds\, dt \\ = \; & \iint_{D'} (x^2, t, 5) \cdot (0,0,1)\, ds\, dt \\ = \; & \iint_{D'} 5\, ds\, dt \\ = \; & 5 \text{ Area}(D') = 20\pi \end{aligned}$$

$$\begin{aligned} & \iint_{\mathbf{X}_2} \mathbf{F} \cdot d\mathbf{S} \\ = \; & \iint_{\mathbf{X}_2} \mathbf{F} \cdot \mathbf{N}\, ds\, dt \\ = \; & \iint_{D'} (s^2, t, s^2+t^2+1) \cdot (-2s, -2t, 1)\, ds\, dt \\ = \; & \iint_{D'} (-2s^3 - 2t^2 + s^2 + t^2 + 1)\, ds\, dt \\ = \; & 4\pi \end{aligned}$$

Since $16\pi = 20\pi - 4\pi$, we have verified Gauss's theorem.

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