Lagrange multipliers
Math 131 Multivariate Calculus
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Constraints and Lagrange multipliers. We've seen how to find extrema for a function when we're looking in an open subset of $\mathbf{R}^n$, namely, find the critical points, then determine which give extrema, perhaps by using the second-derivative test. But what if the extrema occur on the boundaries of that open subset.

For example, suppose f is defined on the closed unit disk, that is, when $\|\mathbf{x}\| \leq 1$, and it's defined by $f(x, y) = 2xy + y^2$. How do you determine if there's an extremum on the boundary? The boundary is the unit circle, which is defined by $\|\mathbf{x}\| = 1$. In this context, the equation $\|\mathbf{x}\| = 1$ is called a constraint when we look for extreme values of $f(\mathbf{x})$ for $\mathbf{x}$ which satisfy the equation. Lagrange developed a technique, now called the method of Lagrange multipliers, to solve this problem.

The method. First, we'll see how it works, then we'll see why it works. Suppose we want to find the extreme values of a function $f : \mathbf{R}^n \to \mathbf{R}$ subject to the constraint $g(\mathbf{x}) = c$, where g is a function $g : \mathbf{R}^n \to \mathbf{R}$ and c is a constant. Introduce a new variable λ, and solve the system of equations

$$\begin{aligned} \nabla f(\mathbf{x}) &= \lambda \nabla g(\mathbf{x}) \\ g(\mathbf{x}) &= c \end{aligned}$$

to get what we call critical points subject to the constraint. Then determine which of these give the extreme values of f.

Example 1. Consider the function $f(\mathbf{x}) = 2xy + y^2$ subject to the constraint $g(x, y) = x^2 + y^2 = 1$. The system mentioned above has these three equations

$$\begin{aligned} f_x(x, y) = 2y &= \lambda g_x(x, y) = \lambda 2x \\ f_y(x, y) = 2x + 2y &= \lambda g_y(x, y) = \lambda 2y \\ g(\mathbf{x}) = x^2 + y^2 &= c = 1 \end{aligned}$$

in the three unknowns x, y, and λ. The first two equations give

$$\lambda = \frac{y}{x} = \frac{x + y}{y} = \frac{x}{y} + 1,$$

and the equation $\lambda = \frac{1}{\lambda} + 1$ has the two solutions $\lambda_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1 - \sqrt{5})$. These lead to the four solutions of the system. Two turn out to be maxima and two minima.

Why it works. Let's look at the planar case. Suppose $\mathbf{x}$ is a point satisfying $g(x, y) = c$. The gradient $\nabla f(\mathbf{x})$ points in the direction in $\mathbf{R}^2$ in which f grows fastest, while $\nabla g(\mathbf{x})$ is normal to the curve $g(x, y) = c$. So, if these directions, $\nabla f(\mathbf{x})$ and $\nabla g(\mathbf{x})$, aren't the same, then f will increase in one direction along the curve and decrease in the opposite direction. A sketch helps here. So the only way an extremum can occur at $\mathbf{x}$ is if $\nabla f(\mathbf{x})$ is some multiple, denoted λ here, of $\nabla g(\mathbf{x})$.

More than one constraint. A variant of this method works when there is more than one constraint. Suppose that $g_1(\mathbf{x}) = c_1$ and $g_2(\mathbf{x}) = c_2$ are two constraints that have to be satisfied. Then solve the system of equations

$$\begin{aligned} \nabla f(\mathbf{x}) &= \lambda_1 \nabla g_1(\mathbf{x}) + \lambda_2 \nabla g_2(\mathbf{x}) \\ g_1(\mathbf{x}) &= c_1 \\ g_2(\mathbf{x}) &= c_2 \end{aligned}$$

to find critical points.

Example 2. Consider the function $f : \mathbf{R}^4 \to \mathbf{R}$ defined by $f(\mathbf{x}) = f(w, x, y, z) = w^2x^2 + y^2z^2$. Find the extrema on the plane of points in $\mathbf{R}^4$ which satisfy

$$\begin{aligned} g_1(\mathbf{x}) = w + x + y + z &= 2 \\ g_2(\mathbf{x}) = w + x - y - z &= 1 \end{aligned}$$

Since $\nabla f(\mathbf{x}) = (2wx^2, 2w^2x, 2yz^2, 2y^2z)$, $\nabla g_1(\mathbf{x}) = (1, 1, 1, 1)$, and $\nabla g_2(\mathbf{x}) = (1, 1, -1, -1)$, we have the system of six equations in six unknowns

$$\begin{array}{rcl} 2wx^2 & = & \lambda_1 + \lambda_2 \\ 2w^x & = & \lambda_1 + \lambda_2 \\ 2yz^2 & = & \lambda_1 - \lambda_2 \\ 2y^z & = & \lambda_1 - \lambda_2 \\ w + x + y + z & = & 2 \\ w + x - y - z & = & 1 \end{array}$$

We can eliminate λ_1 and λ_2 from the first two equations to get the equations $2wx^2 = 2w^2x$ and $2yz^2 = 2y^z$. From the first of these, either $w = 0$ or $x = 0$ or $w = x$. From the second, either $y = 0$ or $z = 0$ or $z = y$. That's nine combination cases to consider in all.

Let's look at just one of them, say $w = 0$ and $z = y$. The last two equations become $x + 2y = 2$ and $x = 1$. In this case we get the unique solution $(w, x, y, z) = (0, 1, \frac{3}{2}, \frac{3}{2})$.

The other eight cases will also yield some solutions. Among all these will be a global minimum, but as f takes arbitrarily large values on this plane, f won't have a maximum.

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