CLARK UNIVERSITY 1887

Line integrals
Math 131 Multivariate Calculus
D Joyce, Spring 2014

Preview. We'll start our study of line integrals. For the next few meetings we'll study line integrals. First, we'll see just what line integrals are, both scalar line integrals and vector line integrals. Second, we'll discuss Green's theorem. Green's theorem relates a double integral over a plane region to a line integral around the boundary of that plane region. Finally, we'll look at conservative vector fields and curls.

Scalar line integrals. So, what is a line integral? It's an integral along a curve. An ordinary integral, $\int_a^b f(s)\,ds$, is the integral along the line segment $[a,b]$. We'll generalize this integral by changing the line segment $[a,b]$ of the x-axis to a segment of any curve in $\mathbf{R}^n$. When we do that, we'll take the integrand $f(\mathbf{x})$ to be a scalar-valued function $\mathbf{R}^n \to \mathbf{R}$. (Soon, we'll look at vector-valued functions, too.)

The natural parameterization of a curve is by its arclength s that we discussed a while ago. For that parameterization, a path $\mathbf{x} : \mathbf{R} \to \mathbf{R}^n$ in $\mathbf{R}^n$ has the property that $\|\mathbf{x}'(s)\| = 1$ for all s. It means that a point is travelling the path at unit speed. At each point $\mathbf{x}(s)$ on the path, the function f has a value, $f(\mathbf{x}(s))$. Just as the ordinary integral, $\int_a^b f(s)\,ds$, sums the values of f along the line segment $[a,b]$ on the x-axis, the line integral, $\int_a^b f(\mathbf{x}(s))\,ds$, sums the values of f along the path $\mathbf{x}(s)$ as s varies from a to b. This line integral is variously denoted

$$\int_{\mathbf{x}} f = \int_{\mathbf{x}} f\,ds = \int_a^b f(\mathbf{x}(s))\,ds.$$

Typically, a path is not given already parametrized by its arclength s. Instead, it is parametrized by a variable t that describes motion along the curve at a variable speed $\|\mathbf{x}'(t)\|$. Rather than reparameterizing the path in terms of s, another way to directly give the line integral is

$$\int_{\mathbf{x}} f\,ds = \int_a^b f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt.$$

Example 1. Evaluate the scalar line integral

$$\int_{\mathbf{x}} (3x + xy + z^3)\,ds$$

where $\mathbf{x}$ is the path $\mathbf{x}(t) = (\cos 4t, \sin 4t, 3t)$ for $t \in [0, 2\pi]$.

Vector line integrals. The vector line integrals we're going to look at are integrals of vectors of a vector field $\mathbf{F}$ dotted with unit tangent vectors $\mathbf{T}$ for the curves. Since the integrand is actually a scalar, the dot product of vectors, the value of these integrals is also a scalar.

A *vector line integral* of a vector-valued function $\mathbf{F} : \mathbf{R}^n \to \mathbf{R}^m$ along a path $\mathbf{x} : [a,b] \to \mathbf{R}^n$ is the integral

$$\int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\,dt$$

which we'll also denote $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$. Note that the $d\mathbf{s}$ in this notation is a vector, not the scalar ds we just used for the scalar line integrals.

These vector line integrals can be given in terms of the unit tangent vector $\mathbf{T}$ of a path. Recall that $\mathbf{T}$ is defined by

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} = \frac{d\mathbf{x}}{ds}.$$

The vector line integral equals

$$\int_{\mathbf{x}} (\mathbf{F} \cdot \mathbf{T})\,ds,$$

and here's the proof.

$$\begin{aligned} & \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\,dt \\ = & \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} \|\mathbf{x}'(t)\|\,dt \\ = & \int_{\mathbf{x}} (\mathbf{F} \cdot \mathbf{T})\,ds \end{aligned}$$

Thus, if we interpret the vector differential $d\mathbf{s}$ as being the product $\mathbf{T}\,ds$, we can literally define the vector line integral as

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F} \cdot \mathbf{T}\,ds.$$

This formulation doesn't depend on the path, that is it doesn't depend on the speed a moving point goes along the curve. But it does depend on which end of the curve it starts as. If you traverse the curve in the other direction, the value of the integral is negated. Thus, the vector line integral depends on the orientation of the curve.

Example 2. Evaluate the vector line integral

$$\int_{\mathbf{x}} (3z, y^2, 6z) \cdot d\mathbf{s}$$

over the path $\mathbf{x}(t) = (\cos t, \sin t, t/3)$ for $t \in [0, \pi]$.

Differential forms. When the integrand $\mathbf{F}$ and the path $\mathbf{x}$ are given by their coordinate functions, vector integrals are written using what are called *differential forms*. Let's stick to $\mathbf{R}^3$ for a while.

Let the vector field $\mathbf{F}$ be given coordinatewise as $\mathbf{F} = (M, N, P)$, that is,

$$\mathbf{F}(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z))$$

and the path $\mathbf{x}$ be given, as usual, as

$$\mathbf{x}(t) = (x(t), y(t), z(t)).$$

Then we can rewrite the vector integral $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$ as

$$\begin{aligned} & \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} \\ = & \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\,dt \\ = & \int_a^b (M\,x'(t) + N\,y'(t) + P\,z'(t))\,dt \\ = & \int_a^b \left(M\,\frac{dx}{dt}\,dt + N\,\frac{dy}{dt}\,dt + P\,\frac{dz}{dt}\,dt \right) \\ = & \int_a^b M\,dx + N\,dy + P\,dz \end{aligned}$$

Here, the differential dx is a notation for $\dfrac{dx}{dt}\,dt$. That last integral is usually abbreviated

$$\int_{\mathbf{x}} M\,dx + N\,dy + P\,dz.$$

The part of the expression after the integral sign, namely,

$$M\,dx + N\,dy + P\,dz,$$

is called a *differential form*. In this course, we're taking differential forms as being just notational devices, but they can be abstracted in a way to make them elements of some abstract algebraic object.

Example 3. Evaluate the line integral

$$\int_C (x^2 - y)\,dx + (x - y^2)\,dy$$

where C is the line segment from $(1, 1)$ to $(3, 5)$. This will require choosing a parametrization of that line segment.

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