Newton's Method
Math 131 Multivariate Calculus
D Joyce, Spring 2014

Newton's method. See also my web site on Newton Basins at `http://aleph0.clarku.edu/~djoyce/newton/newton.html`.

Newton's method for functions of one variable. Newton's method is a method to approximate solutions to equations of the form $f(x) = 0$, that is, how to find roots of functions $f : \mathbf{R} \to \mathbf{R}$. Not only is the method easy to comprehend, it is a very efficient way to find the solution to the equation.

It's an iterative process that starts with a guess x_0, finds a better approximation x_1 based on the value of x_0, then finds an ever better approximation x_2 based on the value of x_1, and continues until the desired accuracy is achieved.

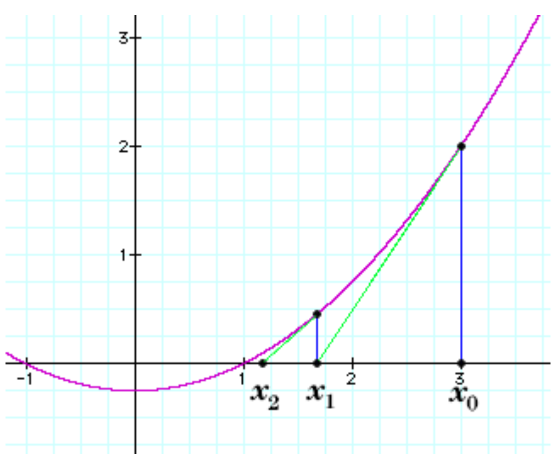

Figure 1: Newton's method for a real function

The process is illustrated in figure 1. Let x_0 be the initial function. Look at the tangent to the graph of the function at the point $(x_0, f(x_0)$ and follow that down to the x-axis to find the next approximation x_1. Algebraically, that means you set x_1 to the value

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Frequently x_1 will be closer to a root than x_0 was. More estimates can be found recursively by

$$x_{k+1} = x_i - \frac{f(x_k)}{f'(x_k)}.$$

The sequence $x_0, x_1, x_2, \ldots, x_k, \ldots$ often approaches a root of the function f.

Newton's method for complex functions. The Newton Basins web site uses a generalization from real variables to complex variables. For that, a function is a complex function $f : \mathbf{C} \to \mathbf{C}$. Derivatives are defined in the same way for a complex function as they are for a real function, so the recursive formula

$$x_{k+1} = x_i - \frac{f(x_k)}{f'(x_k)}$$

makes sense and works for complex functions, too. (For background on complex numbers, see Dave's Short Course on Complex Numbers.)

In particular, the web site examines roots of polynomial functions of complex variables. All the complex numbers that lead to one particular root form what is called a Newton basin for the root.

For example, consider the polynomial with roots i, $-i$, and 1.75,

$$f(z) = (z-i)(z+i)(z-1.75) = z^3 - 1.75z^2 + z - 1.75.$$

Figure 3 shows the three Newton basins for this cubic polynomial. The image is the a square in the complex plane where the real part varies from -2 to 2 and the imaginary part varies from $-2i$ to $2i$. The root i is in the center of the light red circle, the root $-i$ is in the center of the light green circle, and the root 1.75 is in the center of the light blue circle.

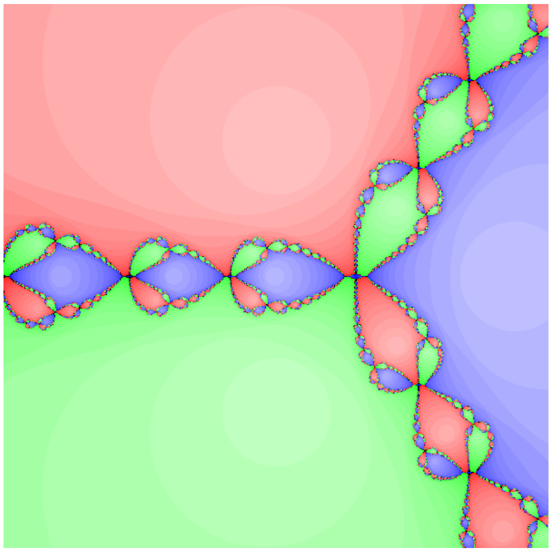

Figure 2: Newton basins

If your first approximation lies fairly near the root $z = i$, then the succeeding approximations approach i very fast. Likewise if the first approximations are near either of the other two roots, the approximations approach that root quickly, too. Note, however, if you start about midway between two of the roots, then you might hop into the main basin of the third root. The boundaries between the three basins are fractals.

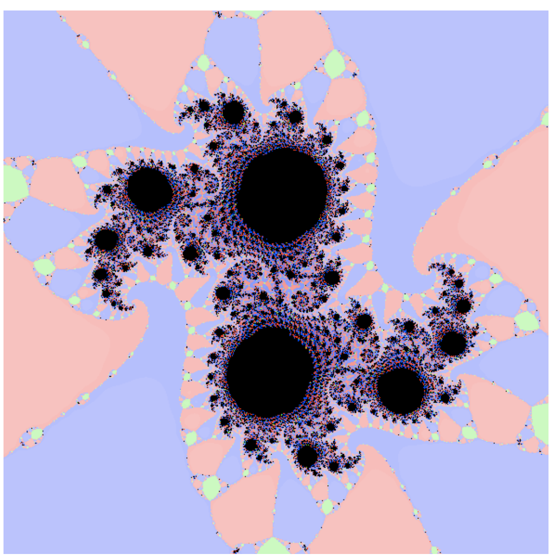

Figure 3: Newton basins for a cubic

Figure 3 shows the Newton basins in small portion of the complex plane for the cubic polynomial with roots $z = 1$, $z = -1.384609 - 0.9i$, and $z = 0.384609 + 0.93i$.

Newton's method for vector fields. In this course we can look at a generalization to vector fields, $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$. With our new concept of derivative, we can write the recursion relation as

$$\mathbf{x}_{k+1} = \mathbf{x}_k - (D\mathbf{f}(\mathbf{x}_k))^{-1}\mathbf{f}(\mathbf{x}_k)$$

where $x \in \mathbf{R}^n$. The total derivative $D\mathbf{f}(\mathbf{x}_k)$ of $\mathbf{f}$ at $\mathbf{x}_k$ is an $n \times n$ matrix, and $(D\mathbf{f}(\mathbf{x}_k))^{-1}$ is the inverse matrix, should the inverse exist.

The text has some examples illustrating how to use this form to solve simultaneous nonlinear equations.

This generalization to multivariate functions includes the previous generalization to complex numbers since a complex function can be interpreted as a function $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^2$.

Math 131 Home Page at
`http://math.clarku.edu/~djoyce/ma131/`